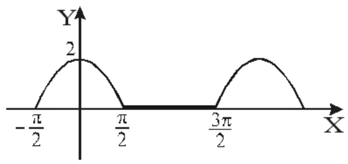
$$f(x) = \cos x + |\cos x| = \begin{cases} 2\cos x & \text{if } 0 \le x < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} \le x \le \pi \end{cases}$$



$$f'\left(\frac{\pi}{2}^-\right) = \lim_{h\to 0} \frac{2\sin h - 0}{h} = -2$$

$$f'\left(\frac{\pi}{2}^+\right) = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

CHAPTER 3

DIFFERENTIABILITY

3.1 INTRODUCTION

Our first application of the concept of limit will be to the problem of determining the instantaneous rate of change of a function. Geometrically, this problem is equivalent to that of finding a tangent line to the graph of the function. Both of these problems are solved by finding the derivative of the given function, as will be described in this chapter.

The Definition of Derivative

Let there be a function y = f(x) defined in a certain interval. The function y = f(x) has a definite value for each value of the argument x in this interval. Let the argument x receive a certain increment Δx (it is immaterial whether it is positive or negative). Then the function y will receive a certain increment Δy.

Thus, for the value of the argument x we will have

$y = f(x)$,

for the value of the argument $x + \Delta x$ we will have

$y + \Delta y = f(x + \Delta x)$.

Let us find the increment of the function Δy

$$\Delta y = f(x + \Delta x) - f(x) \qquad ...(1)$$

Forming the ratio of the increment of the function to the increment of the argument, we get the **difference quotient**

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad ...(2)$$

We then find the limit of this ratio as $\Delta x \to 0$. If this limit exists, it is called the derivative or differential coefficient of the given function f(x) and is denoted $f'(x)$. Thus, by definition,

$$\frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}$$

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad ...(3)$$

Note: Δx is a single symbol and does not mean delta times x. Do not forget that as $\Delta x \to 0$, Δx is getting close to 0, but is not equal to 0.

Consequently, the derivative of a given function y = f(x) with respect to the argument x is the limit of the ratio of the increment in the function Δy to the increment in the argument Δx, when the latter approaches zero, provided the limit exists.

Symbolically, the above equation is written as

$$\frac{dy}{dx} = \frac{df(x)}{dx} = f'(x)$$

where Δx represents finite change and dx represents infinitesimally small change in x.

It is convenient to write Δx as h while calculating the derivative. With such notation the derivative of the function f with respect to the variable x is given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \qquad ...(4)$$

provided the limit exists.

Usually, when we say that a function f has a derivative $f'(x)$ at a point x, it is implied that this derivative is finite, i.e. the above limit is finite. But it may occur that there exists an infinite limit equal to ∞ or –∞. In such cases it is useful to say that the function f has at point x an infinite derivative (equal to ∞ or –∞).

It should be emphasized that although the notation $\frac{df}{dx}$ suggest a ratio, the derivative as we have defined it, is not a ratio – even though it is the limit of one. $\frac{df}{dx}$ is simply an abbreviation of f'.

It will be noted that in the general case, the derivative $f'(x)$ has a definite value for each value of x. If we associate with each value of x the value of the derivative of f(x), if it exists, we obtain a new function of x, called simply the derivative of f(x) with respect to x. This is denoted by $f'(x)$.

The domain of f', the set of points in the domain of f for which the limit (3) exists, may be smaller than the domain of f. If $f'(x)$ exists, we say that f has a derivative (or f is differentiable) at x. $f'(x)$ often has the same domain of definition as f(x), but not always.

The operation of finding the derivative of a function f(x) is called differentiation of the function. If we can find a mathematical expression for $f'(x)$, we can then find the value of the derivative for a particular value of x by mere substitution.

Example 1. Find f' if $f(x) = \frac{1-x}{2+x}$.

Solution $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0}\frac{\frac{1-(x+h)}{2+(x+h)} - \frac{1-x}{2+x}}{h}$$

$$= \lim_{h\to 0}\frac{(1-x-h)(2+x)-(1-x)(2+x+h)}{h(2+x+h)(2+x)}$$

$$= \lim_{h\to 0}\frac{(2-x-2h-x^2-xh)-(2-x+h-x^2-xh)}{h(2+x+h)(2+x)}$$

$$= \lim_{h\to 0}\frac{-3h}{h(2+x+h)(2+x)}$$

$$= \lim_{h\to 0}\frac{-3}{(2+x+h)(2+x)} = -\frac{3}{(2+x)^2}.$$

Example 2. If $f(x) = x^3 - 1$, plot the graph of the derived function f'.

Solution For any real number x,

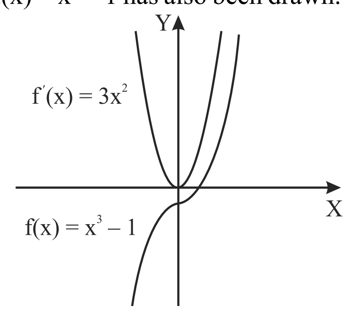
$$f'(x) = \lim_{t\to 0}\frac{f(x+h)-f(x)}{h}.$$

We have $f(x+h) - f(x) = ((x+h)^3 - 1) - (x^3 - 1)$
$= 3x^2h + 3xh^2 + h^3$,

and so $\frac{f(x+h)-f(x)}{h} = 3x^2 + 3xh + h^2$

Consequently, $f'(x) = \lim_{h\to 0}(3x^2 + 3xh + h^2) = 3x^2$.

The graph of the function $f'(x) = 3x^2$ is the parabola shown in figure, on which the graph of the original function $f(x) = x^3 - 1$ has also been drawn.



Here, we considered the derivative of a function f at a general point x. If we change our point of view and let the number x be a fixed number a then we replace x in Equation (4) by a. We obtain

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \quad ...(5)$$

Note that we can also define

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a},$$ provided the limit exists.

Thus, the derivative of a function f is the function f' defined by

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}.$$

Instantaneous Rate of Change–The Physical Meaning of Derivative

Suppose y is a function of x, say y = f(x). Corresponding to a change from x to $x + \Delta x$, the variable y changes from f(x) to $f(x + \Delta x)$.

The change in y is $\Delta y = f(x + \Delta x) - f(x)$, and the average rate of change of y with respect to x over the interval from x to $x + \Delta x$ is

Average rate of change

$$= \frac{\text{change in y}}{\text{change in x}} = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

For example, let $y = f(x) = x^2 + 2x$. Starting at $x_0 = 1$, change x to 1.5. Then $\Delta x = 0.5$. The corresponding change in y is $\Delta y = f(1.5) - f(1) = 5.25 - 3 = 2.25$. Hence the average rate of change of y on the interval between $x = 1$ and $x = 1.5$ is $\frac{\Delta y}{\Delta x} = \frac{2.25}{0.5} = 4.5$.

As the interval over which we are averaging becomes shorter (that is, as $\Delta x \to 0$), the average rate of change approaches what we would intuitively call the instantaneous rate of change of y with respect to x, and the difference quotient approaches the derivative $\frac{dy}{dx}$.

Thus, the instantaneous rate of change at x

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x)$$

provided that the limit exists.

It is conventional to use the word instantaneous even when x does not represent time. The word is, however, frequently omitted. When we say rate of change, we mean instantaneous rate of change.

If the dependence upon time t of the distance s of a moving point is expressed by the formula

$$s = f(t)$$

The velocity v at time t is expressed by the formula

$$v = \lim_{\Delta t \to 0}\frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t}$$

Hence $v = s_t' = f'(t)$

or, the velocity is equal to the derivative of the distance with respect to the time.

In general, suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement (directed distance) of the object from the origin at time t. The function f that describes the motion is called the position function of the object. In the time interval from $t = a$ to $t = a + h$ the change in position is $f(a + h) - f(a)$. The average velocity over this time interval is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h)-f(a)}{h}$$

Now suppose we compute the average velocities over shorter and shorter time intervals $[a, a + h]$. In other words, we let h approach 0. We define the velocity (or instantaneous velocity) v(a) at time $t = a$ to be the limit of these average velocities :

$$v(a) = \lim_{h \to 0}\frac{f(a+h)-f(a)}{h}$$

Example 3. The position of a particle is given by the equation of motion $s = f(t) = 1/(1 + t)$, where t is measured in seconds and s in metres. Find the velocity and the speed after 2 seconds.

Solution The derivative of f when $t = 2$ is

$$f'(2) = \lim_{h \to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h \to 0}\frac{\frac{1}{1+(2+h)} - \frac{1}{1+2}}{h}$$

$$= \lim_{h \to 0}\frac{\frac{1}{3+h} - \frac{1}{3}}{h} = \lim_{h \to 0}\frac{\frac{3-(3+h)}{3(3+h)}}{h}$$

$$= \lim_{h \to 0}\frac{-h}{3(3+h)h} = \lim_{h \to 0}\frac{-1}{3(3+h)} = -\frac{1}{9}.$$

Thus, the velocity after 2 second is $f'(2) = -\frac{1}{9}$ m/s,

and the speed is $|f'(2)| = \left|-\frac{1}{9}\right| = \frac{1}{9}$ m/s.

Example 4. Suppose that a ball is dropped from the top of a tower, 450 m above the ground.

(a) What is the velocity of the ball after 5 seconds?

(b) How fast is the ball travelling when it hits the ground?

Solution We first use the equation of motion

$$s = f(t) = \frac{1}{2}(9.8)\,t^2 = 4.9t^2$$

to find the velocity v(a) after a seconds :

$$v(a) = \lim_{h \to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h \to 0}\frac{4.9(a+h)^2 - 4.9a^2}{h}$$

$$= \lim_{h \to 0}\frac{4.9(a^2+2ah+h^2-a^2)}{h} = \lim_{h \to 0}\frac{4.9(2ah+h^2)}{h}$$

$$= \lim_{h \to 0} 4.9(2a+h) = 9.8a$$

(a) The velocity after 5s is $v(5) = (9.8)(5) = 49$ m/s.

(b) Since the height of tower is 450 m above the ground, the ball will hit the ground at the time a when $s(t_1) = 450$, that is, $4.9a^2 = 450$

This gives $a^2 = \frac{450}{4.9}$ and $a = \sqrt{\frac{450}{4.9}} \approx 9.6$ s.

The velocity of the ball as it hits the ground is therefore

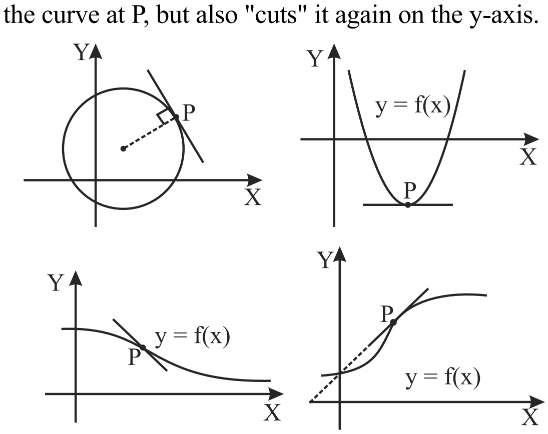
$$v(a) = 9.8a = 9.8\sqrt{\frac{450}{4.9}} \approx 94 \text{ m/s}.$$

Slope of tangent – The Geometrical Meaning of Derivative

The concept of the line tangent to a curve at a point is an important one in geometry. We shall now consider the problem of defining the tangent line to the graph of a function f at a point. We will soon find that the concept of tangent can be expressed in purely analytic terms involving the function f. In fact, the problem leads directly to the definition of the derivative of a function, the central idea in differential calculus.

What does it mean to say that a line is tangent to a curve at a point ? School geometry defines the tangent to a circle as a straight line lying in the plane containing the circle and having a single point in common with it. In other words, the tangent line to a circle at point P is the line that is perpendicular to the radial line at point P, as shown in the first figure below. However, it is not so simple an idea as it may first appear.

For example, how would one define the tangent lines shown in the other figures? We might say that a line is tangent to a curve at point P if it touches, but does not cross, the curve at point P. This definition would work for the first curve shown in the figure, but not for the second. Or we might say that a line is tangent to a curve if the line touches or intersects the curve at exactly one point – this definition would work for a circle but not for more general curves, as the third curve in the figure shows. Here, the tangent "cuts" the curve at the point of tangency instead of "touching" it. In the fourth curve, the tangent "touches" the curve at P, but also "cuts" it again on the y-axis.

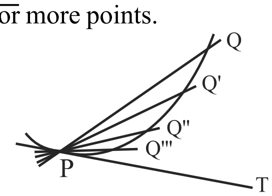


As another example, both x and y-axes have one common point each with the parabola $y = x^2$, but the x-axis is tangent to the parabola, while the y-axis is not. Thus, we see that for more complicated curves, more careful definitions must be made.

If we have a simple continuous curve, our intuition and experience tell us that a point which moves along this curve constantly changes its direction of motion, but that at each point P of the curve there is a definite straight line which gives the direction of motion and closely approximates the curve near P. The line is called the tangent at P and may be approximated in drawing by turning a ruler about P until it appears to have the proper position.

In the figure below, Q, Q′, Q′′, Q′′′ are several successive positions of the point Q as it approaches P. Evidently the secant PQ rotates about P and approaches a limiting position PT, which is the tangent.

Note: A secant line is a line that intersects a curve in two or more points.



Example 5. Find the slope of the secant line joining the points (1, – 2) and (1.2, – 0.56) of the parabola $y = x^2 + 5x - 8$.

Solution Given $y = f(x) = x^2 + 5x - 8$, we find Δy and $\Delta y/\Delta x$ as x changes from $x_0 = 1$ to

$x_1 = x_0 + \Delta x = 1.2$

$\Delta x = x_1 - x_0 = 1.2 - 1 = 0.2$ and

$\Delta y = f(x_0 + \Delta x) - f(x_0) = f(1.2) - f(1)$

$= -0.56 - (-2) = 1.44.$

So $\frac{\Delta y}{\Delta x} = \frac{1.44}{0.2} = 7.2$

The slope of the secant line joining the points (1, – 2) and (1.2, – 0.56) is 7.2.

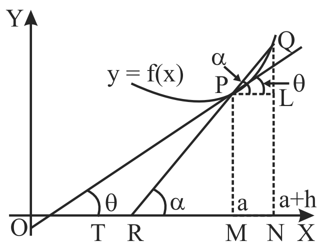
Essentially, the problem of finding the tangent line at a point P boils down to the problem of finding the slope of the tangent line at P. We can approximate this slope using a secant line through the point of tangency and a second point on the curve. This method of approximation leads to the following definition.

Definition The tangent to the curve at the point P is the limit of a secant line through P and another point Q on the curve as Q approaches P.

Let P(a, f(a)) and Q (a + h, f(a + h)) be two points very close to each other on the curve y = f(a). Draw PM and QN perpendiculars from P and Q on x-axis, and draw PL as perpendicular from P on QN. Let the chord PQ produced meet the x-axis at R and ∠QPL = ∠QRN = α. Now in right angled triangle QLP

$$\tan\alpha = \frac{QL}{PL} = \frac{NQ - NL}{MN} = \frac{NQ - MP}{ON - OM}$$

$$= \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h} \qquad ...(1)$$



When h → 0, the point Q moving along the curve tends to P, i.e., Q → P. The chord PQ approaches the tangent line PT at the point P and then α → θ. Now applying

$$\lim_{h\to 0} \tan\alpha = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$\tan\theta = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = m = f'(a)$$

If the limit exists, then the function f(x) is said to have a finite derivative at the point x = a and the line passing through (a, f(a)) with slope m is the tangent line to the graph of f at the point (a, f(a)).

The slope of the tangent line to the graph of f at the point (a, f(a)) is also called the slope of the graph of f at x = a.

If we use the point slope form of the equation of a line, we can write an equation of the tangent line to the curve y = f(x) at the point (a, f(a)) as :

$$y - f(a) = f'(a)(x - a)$$

Further, if $\lim_{x\to a} f'(x)$ does not exist in the ordinary sense, but $\lim_{x\to a} |f'(x)| = \infty$ then the secant line gets steeper and steeper as x approaches a. In this case the line tangent to the graph of the function y = f(x) at the point (a, f(a)) is perpendicular to the x-axis and we call such a line as a **vertical tangent**. The equation of the vertical tangent is x = a. Here, the function f(x) is said to have an infinite derivative at the point x = a.

Note that not every curve has a tangent at any of its points because the existence of tangent depends on the existence of the above limit.

Definition The tangent line to the graph of f at the point P(a,f(a)) is

(i) the line y – f(a) = f′(a)(x – a) through P with slope f′(a), if f′(a) exists;

(ii) the line x = a if $\lim_{x\to a} \left|\frac{f(x) - f(a)}{x - a}\right| = \infty$.

If neither (i) nor (ii) holds, then the graph of f does not have a tangent line at the point P(a, f(a)).

Example 6. Find the equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at the point (3, – 6).

Solution We have $f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$

$$= \lim_{h\to 0} \frac{[(a+h)^2 - 8(a+h) + 9] - [a^2 - 8a + 9]}{h}$$

$$= \lim_{h\to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h}$$

$$= \lim_{h\to 0} \frac{2ah + h^2 - 8h}{h} = \lim_{h\to 0} (2a + h - 8) = 2a - 8.$$

Since the derivative of $f(x) = x^2 - 8x + 9$ at the number a is f′(a) = 2a – 8, the slope of the tangent line at (3, – 6) is f′(3) = 2(3) – 8 = – 2.

Thus, the equation of the tangent line is

$$y - (-6) = (-2)(x - 3) \quad \text{or} \quad y = -2x.$$

Example 7. Let $f(x) = x^2 - 4x + 7$.

(i) Find the average rate of change of f with respect to x between x = 3 and 5.

(ii) Find the instantaneous rate of change of f at x = 3. Interpret the result geometrically.

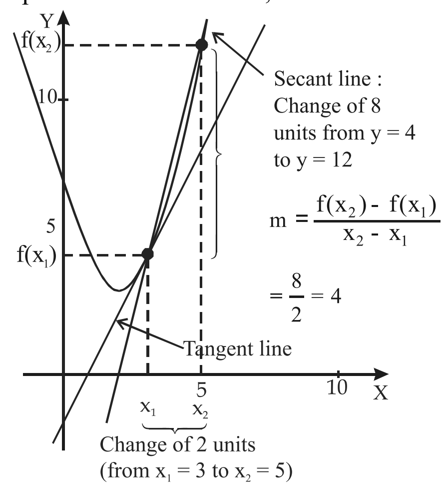
Solution

(i) The (average) rate of change from x = 3 to x = 5 is found by dividing the change in f by the change in x. The change in f from x = 3 to x = 5 is
$f(5) - f(3) = [5^2 - 4(5) + 7] - [3^2 - 4(3) + 7] = 8.$

Thus, the average rate of change is

$$\frac{f(5) - f(3)}{5 - 3} = \frac{8}{2} = 4.$$

The slope of the secant line is 4, as shown in the figure.



(ii) $f'(3) = \lim\limits_{h\to 0}\dfrac{f(3+h)-f(3)}{h}$

$= \lim\limits_{h\to 0}\dfrac{[(3+h)^2-4(3+h)+7]-[3^2-8.3+7]}{h}$

$= \lim\limits_{h\to 0}\dfrac{2h+h^2}{h} = 2.$

The derivative of the function at x = 3 is 2.

Thus, the instantaneous rate of change of f at x = 3 is $f'(3) = 2$.

The tangent line at x = 3 has slope 2, as shown in the figure.

Example 8. Find equation of the tangent line to the graph of the function $f(x) = \sqrt[3]{x}$ at the origin.

Solution We have

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{\sqrt[3]{x}}{x} = \lim_{x\to 0}\frac{1}{x^{2/3}} = \infty.$$

Therefore, the tangent is the vertical line x = 0.

Example 9. Show that the following functions do not have finite derivative at the indicated points :

(a) $y = \sqrt[5]{x^2}$ at $x = 0$ (b) $y = \sqrt[5]{x-1}$ at $x = 1$

Solution (a) $f'(0) = \lim\limits_{\Delta x\to 0}\dfrac{\sqrt[5]{(\Delta x)^2}}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{1}{\sqrt[5]{(\Delta x)^3}}$.

Now the left hand limit is $-\infty$ and right hand limit is ∞. Hence, the limit does not exists and there is no derivative at x = 0.

(b) $f'(1) = \lim\limits_{\Delta x\to 0}\dfrac{\sqrt[5]{1+\Delta x-1}}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{1}{\sqrt[5]{(\Delta x)^4}} = \infty.$

Essentially, both the functions donot have a finite derivative at the given points.

Meaning of Sign of Derivative

Remember that a positive derivative signals an increasing quantity and that a negative derivative signals a decreasing quantity.

This is obvious also from the geometrical interpretation of $\dfrac{dy}{dx}$. For, if x and y are increasing together $\dfrac{dy}{dx}$ is the tangent of an acute angle and therefore positive, while if, as x increases y decreases, $\dfrac{dy}{dx}$ represents the tangent of an obtuse angle and is negative.

This aspect will be studied in detail in the chapter of monotonicity.

3.2 DIFFERENTIABILITY

The derivative of the function f with respect to the variable x at x = a is given by

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

provided the limit exists.

If the derivative of f(x) exists for x = a, we say that f(x) is differentiable (derivable) for x = a, otherwise, it is non-differentiable.

If a function f(x) is differentiable at x = a, the graph of f(x) will be such that there is a tangent with finite slope to the graph at the corresponding point. But if f(x) is non-differentiable at x = a, there will be either vertical tangent or no tangent with finite slope at the corresponding point of the graph.

If f(x) is differentiable for every value of x in an interval I, we say that f(x) is differentiable in the interval I.

If f is differentiable at every point in its domain, it is simply called a differentiable function.

If f is differentiable on $(-\infty, \infty)$ we will say that f is differentiable everywhere.

Example 1. If g(x) = 1/x, find the derivative $g'(x)$. Also find the domain of $g'(x)$.

Solution $g'(x) = \lim\limits_{h\to 0}\dfrac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h}$

$$= \lim_{h\to 0}\frac{x-(x+h)}{hx(x+h)} \qquad = \lim_{h\to 0}\frac{-1}{x(x+h)}$$

$$= -\frac{1}{x^2}$$

Evidently, the domain of $g'(x)$ is the set of all non-zero numbers, just as is the domain of $g(x)$. That is, g is differentiable for all x except $x = 0$.

One-sided derivatives

For the derivative of f at a point x to exist, it is necessary that the function f be defined in a certain neighbourhood of the point x including the point x itself. Then the expression $\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$ is defined for all sufficiently small Δx different from zero.

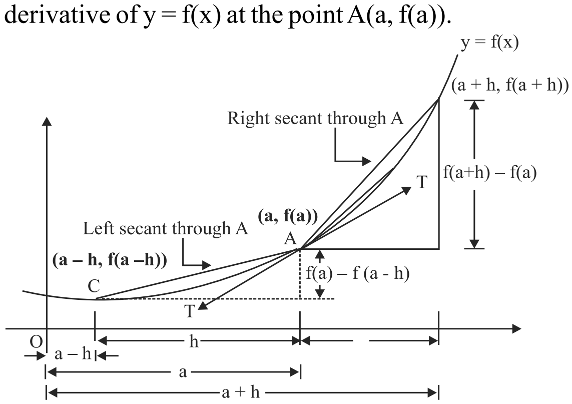
Now, the limit $f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$ does not always exist for any function f defined in a neighbourhood of the point x.

The one-sided limit form of the derivative is useful in investigating the existence of derivative.

If in the above limit it is assumed that Δx tends to zero attaining only positive values ($\Delta x > 0$), then the corresponding limit (wherever it exists) is called the right hand derivative of the function f at a point x. We shall denote it as $f'(x^+)$.

Analogously, the limit, when Δx tends to zero running through negative values ($\Delta x < 0$), is termed as the left hand derivative of f at x (denoted as $f'(x^-)$.

Of course, to compute $f'(x^+)$ (or $f'(x^-)$) it is only necessary that the function f be defined at the point x and on the right of it in a certain neighbourhood (or at x and on the left of x). Consider the existence of derivative of $y = f(x)$ at the point A(a, f(a)).



Let us choose a point B to the right of A. If the slope of tangent at A is evaluated by allowing B to approach A along the curve, then we have

$$\text{Slope of right tangent} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = f'(a^+)$$

which is called the **Right Hand Derivative (R.H.D.)** of f at x, provided the limit exists..

Let us choose a point C to the left of A. If the slope of tangent at A is evaluated by allowing C to approach P along the curve, then we have

$$\text{Slope of left tangent} = \lim_{h\to 0}\frac{f(a)-f(a-h)}{h} = f'(a^-)$$

which is called the **Left Hand Derivative (L.H.D.)** of f at x, provided the limit exists..

Sometimes, we prefer to write the left hand derivative as $f'(a^-) = \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$.

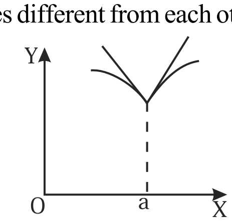
Differentiability at an Interior Point

A function f(x) is said to be differentiable at some interior point x = a in its domain if both L.H.D. and R.H.D. exist at that point, and are equal, that is, if $f'(a^+) = f'(a^-)$ = a finite quantity.

This geometrically means that a unique tangent with finite slope can be drawn at x=a as shown in the figure.

There are cases of failure of existence of $f'(a)$ and even of $f'(a^-)$ and $f'(a^+)$ at the point x = a, i.e. when the graph of the function has neither a right, nor a left tangent at the given point.

It may happen that f has at x = a, left and right derivatives different from each other : i.e. $f'(a^-) \neq f'(a^+)$.



Then such a point is called a **sharp corner**. In this case the tangent line does not exist at the point, but we say that there exist right and left tangents with different slopes at that point.

Differentiability at Endpoints

At the left endpoint of domain, the existence of R.H.D. is a sufficient condition for differentiability and at the right endpoint of domain, the existence of L.H.D. is a sufficient condition for differentiability.

If x = a is the left endpoint of the domain of the function f, then f is differentiable at x = a if the right hand derivative

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \text{ exists.}$$

If x = b is the right endpoint of the domain of the function f, then f is differentiable at x = b if the left hand derivative

$$\lim_{h\to 0}\frac{f(b-h)-f(b)}{-h} \text{ exists.}$$

Example 2. Examine the differentiability of the function $f(x) = ln^2 x$ at x = 1.

Solution $\text{L.H.D.} = f'(1^-) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$

$$= \lim_{h\to 0}\frac{\ln^2(1-h)-0}{-h} = 0.$$

$$\text{R.H.D.} = f(1^+) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{\ln^2(1+h)-0}{h} = 0.$$

Since L.H.D.= R.H.D. f(x) is differentiable at x = 1 and $f'(1) = 0$.

Example 3. Examine the differentiability of the function $f(x) = |ln\, x|$ at x = 1.

Solution $f(x) = |ln\, x| = \begin{cases} \ln x & \text{if } \ln x \ge 0 \\ -\ln x & \text{if } \ln x < 0 \end{cases}$

$$= \begin{cases} \ln x & \text{if } x \ge 1 \\ -\ln x & \text{if } 0 < x < 1 \end{cases}$$

$$\text{L.H.D.} = f'(1^-) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0}\frac{-\ln(1-h)-0}{-h} = -1.$$

$$\text{R.H.D.} = f(1^+) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{\ln(1+h)-0}{h} = 1.$$

Since L.H.D. ≠ R.H.D., f(x) is not differentiable at x=1.

Example 4. Let $f(x) = \begin{cases} \dfrac{\sin x^2}{x} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$

Find the slope of tangent at x = 0, if it exists.

Solution $\text{L.H.D.} = f'(0^-) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$

$$= \lim_{h\to 0}\frac{\sin h^2}{+h^2} - 0 = 1.$$

$$\text{R.H.D.} = f'(0^+) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{\sin h^2}{h^2} - 0 = 1.$$

Since L.H.D.= R.H.D., f(x) is differentiable at x = 0 and the slope of tangent at x = 0 is 1.

Example 5. Comment on the differentiability of

$$f(x) = \begin{cases} x, & x < 1 \\ x^2, & x \ge 1 \end{cases} \text{ at } x = 1.$$

Solution $\text{L.H.D.} = f'(1^-) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$

$$= \lim_{h\to 0}\frac{1-h-1}{-h} = 1.$$

$$\text{R.H.D.} = f'(1^+) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{(1+h)^2-1}{h} = \lim_{h\to 0}\frac{1+h^2+2h-1}{h}$$

$$= \lim_{h\to 0}(h+2) = 2.$$

Since L.H.D. ≠ R.H.D., f(x) is not differentiable at x = 1.

Example 6. If $f(x) = A + Bx^2,\ x < 1$

$= 3Ax - B + 2,\quad x \ge 1$

then find A and B so that f(x) becomes differentiable at x = 1.

Solution $\text{L.H.D.} = f'(1^-) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$

$$= \lim_{h\to 0}\frac{A+B(1-h)^2-3A+B-2}{-h}$$

$$= \lim_{h\to 0}\frac{(-2A+2B-2)+Bh^2-2Bh}{-h}$$

Hence, for this limit to exist $-2A+2B-2=0$

i.e., B = A + 1. Then,

$$f'(1^-) = \lim_{h\to 0} (Bh - 2B) = 2B$$

$$\text{R.H.D.} = f'(1^+) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0} \frac{3A(1+h) - B + 2 - 3A + B - 2}{h}$$

$$= \lim_{h\to 0} \frac{3Ah}{h} = 3A.$$

Since f is differentiable at $x = 1$, we have $f'(1^-) = f(1^-)$

$\Rightarrow \quad 3A = 2B \Rightarrow 3A = 2(A+1)$

$\Rightarrow \quad A = 2, B = 3.$

Example 7. Let $f(x) = \begin{cases} [\cos \pi x] & x \le 1 \\ 2\{x\} - 1 & x > 1 \end{cases}$. Comment on the derivability at $x = 4$.

Solution $f(1^-) = \lim_{h\to 0} \frac{f(1-h) - f(1)}{-h}$

$$= \lim_{h\to 0} \frac{\cos(\pi - \pi h) + 3}{-h} = \lim_{h\to 0} \frac{1 - \cos \pi h}{-h}$$

$$= \lim_{h\to 0} \frac{2\sin^2 \frac{h}{2}}{-h} = 0.$$

$$f(1^+) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0} \frac{2\{1+h\} + 1 - 1}{h} = \lim_{h\to 0} 2h = 2.$$

Since L.H.D. ≠ R.H.D., f(x) is not derivable at $x = 3$.

Example 8. A function f(x) is such that

$$f\left(x + \frac{\pi}{2}\right) = \frac{\pi}{2} - |x| \ \forall x. \text{ Find } f'\left(\frac{\pi}{2}\right), \text{ if its exists.}$$

Solution Given that $f\left(x + \frac{\pi}{2}\right) = \frac{\pi}{2} - |x|$

$$f'\left(\frac{\pi^+}{2}\right) = \lim_{h\to 0} \frac{f\left(\frac{\pi}{2} + h\right) - f\left(\frac{\pi}{2}\right)}{h} = \frac{\frac{\pi}{2} - |h| - \frac{\pi}{2}}{h} = -1.$$

$$f'\left(\frac{\pi^-}{2}\right) = \lim_{h\to 0} \frac{f\left(\frac{\pi}{2} - h\right) - f\left(\frac{\pi}{2}\right)}{-h} = \frac{\frac{\pi}{2} - |-h| - \frac{\pi}{2}}{-h} = 1.$$

$\Rightarrow \quad f'\left(\frac{\pi}{2}\right)$ does not exist.

Example 9. Show that the function $f(x) = |\cos x|$ does not have derivative at $x = \frac{2k+1}{2}\pi, k \in I.$

Solution $f'\left(\frac{2k+1}{2}\pi^-\right) = \lim_{\Delta x\to 0^-} \frac{\left|\cos\left(\frac{2k+1}{2}\pi + \Delta x\right)\right|}{\Delta x}$

$$= \lim_{\Delta x\to 0^-} \frac{|\sin \Delta x|}{\Delta x} = -1$$

$$f'\left(\frac{2k+1}{2}\pi^+\right) = \lim_{\Delta x\to 0^+} \frac{|\sin \Delta x|}{\Delta x} = 1.$$

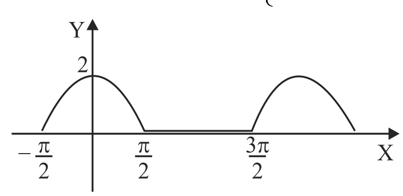
Since L.H.D. ≠ R.H.D., f(x) is not derivable at

$$x = \frac{2k+1}{2}\pi, k \in I.$$

Example 10. Examine the differentiability of the function $f(x) = \cos x + |\cos x|$ at $x = \frac{\pi}{2}$.

Solution

$$f(x) = \cos x + |\cos x| = \begin{cases} 2\cos x & \text{if } 0 \le x < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} \le x \le \pi \end{cases}$$



$$f'\left(\frac{\pi}{2}^-\right) = \lim_{h\to 0} \frac{2\sin h - 0}{h} = -2$$

$$f'\left(\frac{\pi}{2}^+\right) = \lim_{h\to 0} \frac{0 - 0}{h} = 0.$$

Since the left and right derivatives are different from each other, the graph of the function has a corner point at $x = \frac{\pi}{2}$ and f(x) is not differentiable here.

Example 11. Given $f(x) = [x] \tan(\pi x)$ where [.] denotes greatest integer function, find the LHD and RHD at $x = k$, where $k \in I$.

Solution $f(x) = [x] \tan(\pi x)$

$$f'(k^+) = \lim_{h\to 0} \frac{f(k+h) - f(k)}{h}$$

$$= \lim_{h\to 0} \frac{[k+h]\cdot \tan(k\pi + \pi h) - 0}{h}$$

$$= \lim_{h\to 0} \frac{\pi k \tan(\pi h)}{\pi h} = k\pi$$

$$f'(k^-) = \lim_{h\to 0} \frac{f(k-h) - f(k)}{-h}$$

$$= \lim_{h\to 0} \frac{[k-h]\cdot \tan(k\pi - \pi h) - 0}{-h}$$

$$= \lim_{h\to 0} \frac{-(k-1)\tan(\pi h)}{-h}$$

$$= \pi(k-1).$$

Hence R.H.D. $= f(k^+) = k\pi$ and L.H.D. $= f'(k^-) = (k-1)\pi$.

Example 12.

$$\text{Let } g(x) = \begin{cases} \dfrac{x^2 + x\tan x - x\tan 2x}{ax + \tan x - \tan 3x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

If $g'(0)$ exists and is equal to a non-zero value b, then find the value of $\frac{b}{a}$.

Solution $g'(0) = b = \lim_{x\to 0} \frac{x^2 + x\tan x - x\tan 2x}{x(ax + \tan x - \tan 3x)}$

$$= \lim_{x\to 0} \frac{x + \tan x - \tan 2x}{ax + \tan x - \tan 3x}$$

$$= \lim_{x\to 0} \frac{x + \left(x + \frac{x^3}{3} + \frac{2}{15}x^5 + ...\right) - \left(2x + \frac{8x^3}{3} + \frac{2}{15}\cdot 32x^5 + ...\right)}{ax + \left(x + \frac{x^3}{3} + \frac{2}{15}x^5 + ...\right) - \left(3x + \frac{27x^3}{3} + \frac{2}{15}243x^5 + ...\right)}$$

$$= \lim_{x\to 0} \frac{x^3\left(-\frac{7}{3} + \frac{-62}{15}x^2 + ...\right)}{(a+1-3)x + \left(\frac{1}{3} - 9\right)x^3 + \frac{2}{15}(-242)x^5 + ...}$$

limit b can be non - zero if $a + 1 - 3 = 0$

$$\therefore\ a = 2 \text{ and hence } b = \frac{-\frac{7}{3}}{\frac{1}{3} - 9} = \left(\frac{-7}{3}\right)\left(\frac{3}{-26}\right)$$

$$= \frac{7}{26} \Rightarrow \frac{b}{a} = \frac{7}{52}.$$

Example 13. Let $f(x) = \begin{cases} ax^2 - bx + 2 & \text{if } x < 3 \\ bx^2 - 3 & \text{if } x \geq 3 \end{cases}$.

Find the values of a and b so that f(x) is differentiable everywhere.

Solution $f'(3^-)$ and $f'(3^+)$ exist finitely.

$$\Rightarrow\ f'(3^-) = \lim_{h\to 0} \frac{f(3-h) - f(3)}{-h}$$

$$= \lim_{h\to 0} \frac{a(3-h)^2 - b(3-h) + 2 - (9b-3)}{-h}$$

$$= \lim_{h\to 0} \frac{ah^2 + h(b-6a) + 9a - 12b + 5}{(-h)} \quad ...(1)$$

$\because$ $f'(3^-)$ exists, the numerator must tend to zero as $h \to 0$

$$\therefore\ 9a - 12b + 5 = 0 \quad ...(2)$$

Now from (1) and (2)

$$f'(3^-) = \lim_{h\to 0} \frac{ah^2 + h(b-6a)}{-h} = 6a - b,$$

and $f'(3^+) = \lim_{h\to 0} \frac{f(3-h) - f(3)}{h}$

$$= \lim_{h\to 0} \frac{b(3+h)^2 - 3 - (9b-3)}{h}$$

$$= b \lim_{h\to 0} \frac{(3+h)^2 - 9}{h} = 6b. \quad ...(3)$$

Given $f'(3^-) = f'(3^+)$

$$\Rightarrow\ 6a - b = 6b \Rightarrow b = \frac{6a}{7}. \quad ...(4)$$

From (2) and (4),

$$9a - \frac{72a}{7} + 5 = 0 \quad \Rightarrow -\frac{9a}{7} + 5 = 0$$

$$\therefore\ a = \frac{35}{9},\ b = \frac{10}{3}.$$

Example 14. A function f is defined by $f(x^2) = x^3$ for all $x > 0$. Show that f is differentiable at 4.

Solution

$$f'(4) = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \frac{f\left((\sqrt{4+h})^2\right) - f\left(2^2\right)}{h}$$

$$= \lim_{h\to 0}\frac{(4+h)^{3/2}-8}{h} = \lim_{h\to 0}\frac{8[(1+h/4)^{3/2}-1]}{h}$$

$$= \lim_{h\to 0}\frac{8\left[1+\frac{3}{2}\frac{h}{4}+....-1\right]}{h} = \lim_{h\to 0}\frac{8\left[\frac{3}{8}h+\frac{3}{8}\left(\frac{h^2}{16}\right)+...\right]}{h}$$

$= 3.$

Hence f is differentiable at 4.

Example 15. Let f be differentiable at $x = a$ and let $f(a) \neq 0$. Evaluate $\lim_{n\to\infty}\left\{\frac{f(a+1/n)}{f(a)}\right\}^n$.

Solution $l = \lim_{n\to\infty}\left\{\frac{f(a+1/n)}{f(a)}\right\}^n$ (1^∞ form)

$$l = (\text{exp.})\left(\lim_{n\to\infty} n\left\{\frac{f(a+1/n)-f(a)}{f(a)}\right\}\right)$$

put $n = \frac{1}{h}$

$$= (\text{exp.})\left(\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\cdot\frac{1}{f(a)}\right)$$

$$= (\text{exp.})\left(\frac{f'(a)}{f(a)}\right) = e^{\frac{f'(a)}{f(a)}}.$$

Example 16. Let $f : R \to R$ satisfying $|f(x)| \le x^2$, $\forall\ x \in R$ then show f(x) is differentiable at $x = 0$.

Solution Since, $|f(x)| \le x^2$, $\forall\ x \in R$

at $x = 0$, $|f(0)| \le 0 \Rightarrow f(0) = 0$ ...(1)

$\therefore$ $f'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)}{h}$ ....(2)

{f(0) = 0 from (1)}

Now, $\left|\frac{f(h)}{h}\right| \le |h| \Rightarrow -|h| \le \frac{f(h)}{h} \le |h|$

$\Rightarrow \lim_{h\to 0}\frac{f(h)}{h} = 0$ ...(3)

{using Sandwitch theorem}

$\therefore$ from (2) and (3), we get $f'(0) = 0$.

i.e. f(x) is differentiable at $x = 0$.

Concept Problems A

1. Examine the differentiability of the function $f(x) = e^{-|x|}$ at $x = 0$.
2. Let $f(x) = \begin{cases}\sin x & \text{if } x < \pi \\ mx + n & \text{if } x \ge \pi\end{cases}$ where m and n are constants. Determine m and n such that f is derivable at $x = \pi$.
3. Discuss the differentiability of f at $x = 1$
   $f(x) = 3^x, \quad -1 \le x \le 1$
   $= 4 - x, \quad 1 < x < 4$
4. (a) If $g(x) = x^{2/3}$, show that $g'(0)$ does not exist.
   (b) If $a \neq 0$, find $g'(a)$.
   (c) Show that $y=x^{2/3}$ has a vertical tangent line at(0, 0).
   (d) Illustrate part (c) by graphing $y = x^{2/3}$.
5. Compute the difference quotient for the function defined by $f(x) = \begin{cases}\frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0\end{cases}$
   Do you think f(x) is differentiable at x = 0? If so, what is the equation of the tangent line at x = 0 ?
6. Let $f(x) = \begin{cases} x\tan^{-1} 1/x, & x \neq 0 \\ 0 \quad , & x = 0\end{cases}$.
   Comment on the derivative of f(x) at $x = 0$.

Practice Problems A

7. Find the slope of the secant to the parabola $y = 2x - x^2$, if the abscissas of the points of intersection are equal to : $x_1 = 1$, $x_2 = 1 + h$. To what limit does the slope of the secant tend if $h \to 0$?
8. Find $f'(0^+)$ and $f'(0^-)$ if
   (a) $f(x) = \sqrt{\sin(x^2)}$
   (b) $f(x) = \sin^{-1}\frac{a^2 - x^2}{a^2 + x^2}$

(c) $f(x) = \dfrac{x}{1+e^{\frac{1}{x}}}$, $x \neq 0$; $f(0) = 0$

(d) $f(x) = x^2 \sin \dfrac{1}{x}$, $x \neq 0$; $f(0) = 0$.

9. Discuss the differentiability of the function

$$f(x) = \frac{x(3e^{1/x}+4)}{2-e^{1/x}}, \quad x \neq 0$$
$$= 0, \quad x = 0$$

at $x = 0$.

10. Prove that the function

$$f(x) = \begin{cases} |x|^{3/2} \sin \dfrac{1}{x} & \text{if } x \neq 0, x \in R, \\ 0 & \text{if } x = 0, \end{cases}$$

is differentiable at the point $x = 0$, and $f'(0) = 0$.

11. It is known that $f(0) = 0$ and there exists a limit of the expression $\dfrac{f(x)}{x}$ as $x \to 0$. Prove that the limit is equal to $f'(0)$.

12. Prove that if $f(x)$ and $\varphi(x)$ are equal to zero for $x = 0$ and have derivatives at $x = 0$, $\varphi'(0)$ being not equal to zero, then $\lim\limits_{x\to 0} \dfrac{f(x)}{\varphi(x)} = \dfrac{f'(0)}{\varphi'(0)}$.

13. Prove that if $f(x)$ has a derivative at $x = a$, then $\lim\limits_{x\to a} \dfrac{xf(a) - af(x)}{x-a} = f(a) - af'(a)$.

14. Let $f(x)$ be an even function defined for $x \in R$. If $f'(0)$ exists, find its value.

15. Let $f(x) = \dfrac{x\,g(x)}{|x|}$, $g(0) = g'(0) = 0$ and $f(x)$ be continuous at $x = 0$. Find $f'(0)$, if it exists.

3.3 REASONS OF NON-DIFFERENTIABILITY

We now find the reasons because of which a function does not have a derivative at a point.

A function has a derivative at a point $x = a$ if the slopes of the secant lines through $P(a, f(a))$ and a nearby point Q on the graph approach a limit, as Q approaches P. However, a function will fail to have a derivative at a point where the graph has

(i) a corner, where the one-sided derivatives differ.

(ii) an oscillation point, where the one-sided derivative(s) does (do) not exist.

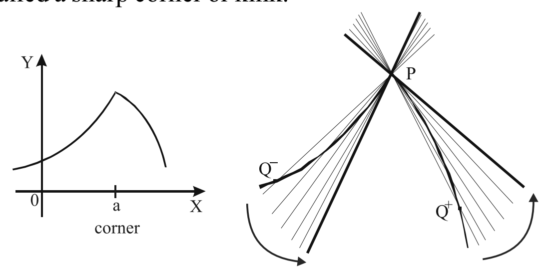
(iii) a vertical tangent, where the absolute value of slope of PQ approaches ∞.

(iv) a discontinuity.

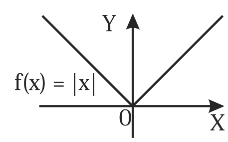
Such slopeless points may be roughly classified as "sharp points", "oscillation points", "steep points", and "points of discontinuity".

CASE I : Sharp corner or kink

If it happens that in a continuous function f at $x = a$, the left hand and right hand derivatives are different from each other i.e. $f'(a^-) \neq f'(a^+)$, then such a point is called a sharp corner or kink.



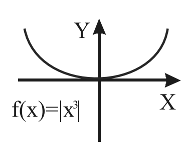
For example, we see that the function $y = |x|$ as shown in the figure has a graph which changes its direction abruptly at $x = 0$. The point $x = 0$ is called a corner and the function $y = |x|$ is not differentiable at 0.



In general, if the graph of a function f has a "corner" or "kink" in it, then the graph of f has no tangent at this point and f is not differentiable there. This can be verified by computing the left and right hand derivatives at the point.

CAUTION

But one should be careful about predicting a corner at a point. For instance, one may feel that the function $f(x) = |x^3|$ has a corner at $x = 0$ and hence non-differentiable there. This is found to be wrong.



On computing, we find that $f'(a^-) = 0 = f'(a^+)$.
So, f is differentiable at $x = 0$.

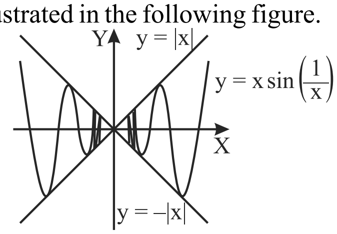
CASE II : Oscillation point

If in a continuous function f, either the left hand derivative or right hand derivative does not exist at x = a, because of too much oscillation of the graph in the neighbourhood of the point, then such a point is called an oscillation point.

For example, the function $f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

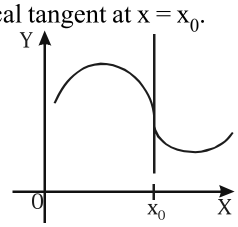
has an oscillation point at x = 0.

This is illustrated in the following figure.



CASE III : Vertical Tangent

A continuous function may also fail to be differentiable at $x = x_0$ if the absolute value of its difference quotient approaches infinity. In this case, the function is said to have a vertical tangent at $x = x_0$.



Vertical Tangent

Definition The curve y = f(x) has a vertical tangent line at the point $(x_0, f(x_0))$ provided that f is continuous at a and

$$|f'(x)| \to \infty \text{ as } x \to x_0 \quad \text{...(1)}$$

Note that the requirement that f be continuous at $x = x_0$ implies that $f(x_0)$ must be defined. Thus, it would be pointless to ask about a line (vertical or not) tangent to the curve y = 1/x where x = 0.

If f is defined on only one side of $x = x_0$, we mean by (1) that $|f'(x)| \to \infty$ as x approaches x_0 from that side.

For example, the function $f(x) = x^{1/3}$ is continuous at x = 0, and

$$\text{L.H.D.} = \lim_{h\to 0} \frac{(0-h)^{1/3} - 0}{-h} = \infty$$

$$\text{R.H.D} = \lim_{h\to 0} \frac{(0+h)^{1/3} - 0}{h} = \infty$$

Since both L.H.D. and R.H.D are infinite, the function has a vertical tangent at x = 0.

∴ The y-axis i.e. x = 0 is the vertical tangent to $f(x) = x^{1/3}$ at origin. Note that f is not differentiable at x = 0.

The graph of a continuous function f having a vertical tangent at $(x_0, f(x_0))$ can have

(a) $f'(x)$ approaching either ∞ or $-\infty$ as $x \to x_0^-$ and as $x \to x_0^+$, i.e. where the slope of the secant line approaches ∞ from both sides or approaches $-\infty$ from both sides, or,

(b) $f'(x)$ approaching ∞ from one side and $-\infty$ from the other side. In such a case, the function f is said to have a **cusp** at x_0.

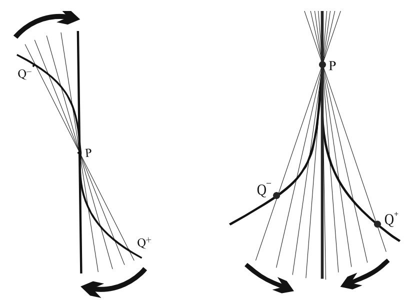


Figure (a) Figure (b)

In figure (a), the slope of PQ approaches $-\infty$ from both sides. In figure (b), the slope of PQ approaches ∞ from left side and $-\infty$ from right side.

Thus, the graph of the continuous function $f(x) = x^{2/3}$ has a vertical tangent line at the origin, even though f is not differentiable at x = 0, since, the function $f(x) = x^{2/3}$ is continuous at x = 0, and has infinite derivative at x = 0:

$$\text{L.H.D.} = \lim_{h\to 0} \frac{(0-h)^{2/3} - 0}{-h} = -\infty$$

$$\text{R.H.D} = \lim_{h\to 0} \frac{(0+h)^{2/3} - 0}{h} = \infty.$$

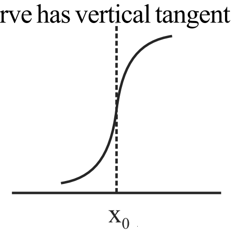
Further, since the one-sided derivatives are infinity opposite signs, we say that its graph has a cusp at x = 0.

The graph of the function $f(x) = 1-\sqrt[5]{x^2}$, has a cusp"(rather than a corner) at the point (0,1) because the derivative $f'(x) = -\frac{2}{5}x^{-3/5}$ approaches ∞ as $x \to 0^-$ and approaches $-\infty$ as $x \to 0^+$. The curve also has a vertical tangent at the point.

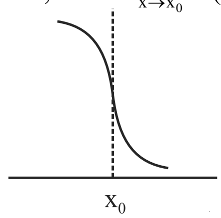
Note: A function cannot have a vertical tangent at a point of discontinuity. For example,

$y = \text{sgn}(x)$ does not have a vertical tangent at $x = 0$ even if both the one- sided derivatives at $x = 0$ are infinite.

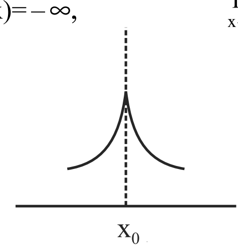
The figures give below show four curve elements that are commonly found in graphs of functions that involve radicals or fractional exponents. In all four cases, the function is not differentiable at x_0 because the secant line through $(x_0, f(x_0))$ and $(x, f(x))$ approaches a vertical position as x approaches x_0 from either side. Thus, in each case, the curve has vertical tangent line at $(x_0, f(x_0))$.



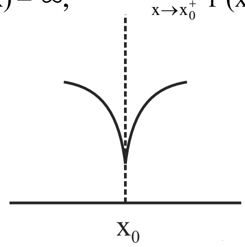
$\lim\limits_{x\to x_0^-} f'(x) = \infty$, $\quad \lim\limits_{x\to x_0^+} f'(x) = \infty$



$\lim\limits_{x\to x_0^-} f'(x) = -\infty$, $\quad \lim\limits_{x\to x_0^+} f'(x) = -\infty$



$\lim\limits_{x\to x_0^-} f'(x) = \infty$, $\quad \lim\limits_{x\to x_0^+} f'(x) = -\infty$



$\lim\limits_{x\to x_0^-} f'(x) = -\infty$, $\quad \lim\limits_{x\to x_0^+} f'(x) = \infty$

Note that the third and fourth figures represent a cusp.

Consider $f(x) = (x-4)^{2/3}$.

$$f'(x) = \frac{2}{3}(x-4)^{1/3} = \frac{2}{3(x-4)^{1/3}}$$

There is a vertical tangent and cusp at $x = 4$ since $f(x) = (x-4)^{2/3}$ is continuous at $x = 4$ and

$$\lim_{x\to 4^+} f'(x) = \lim_{x\to 4^+} \frac{2}{3(x-4)^{1/3}} = \infty$$

$$\lim_{x\to 4^-} f'(x) = \lim_{x\to 4^-} \frac{2}{3(x-4)^{1/3}} = -\infty.$$

Now consider $f(x) = 6x^{1/3} + 3x^{4/3}$.

$$f'(x) = 2x^{-2/3} + 4x^{1/3} = 2x^{-2/3}(1+2x) = \frac{2(2x+1)}{x^{2/3}}.$$

There is point of vertical tangency at $x = 0$, since

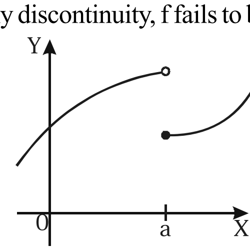
$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} \frac{2(2x+1)}{x^{2/3}} = \infty$$

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} \frac{2(2x+1)}{x^{2/3}} = \infty.$$

It does not have a cusp at $x = 0$.

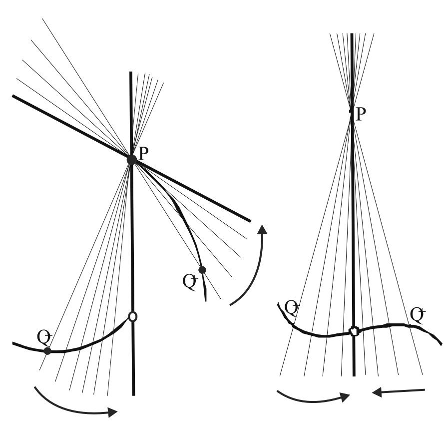
CASE IV : Discontinuity

There is another way for a function not to have a derivative. We shall prove in the next section that if f is discontinuous at $x = a$, then f is not differentiable at $x = a$. So, at any discontinuity, f fails to be differentiable.



Discontinuity

The following figures of discontinuous functions suggest that the secant lines donot approach a finite slope from atleast one side.



3.4 RELATION BETWEEN CONTINUITY AND DIFFERENTIABILITY

Theorem If a function f is differentiable at x = a then it must be continuous at x = a.

Proof Since f is differentiable at x = a,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \text{ exists.} \quad ...(1)$$

To prove that f is continuous at x = a, we must prove that $\lim_{h\to 0}\left(f(a+h) - f(a)\right) = 0$.

$$\text{Now } \lim_{h\to 0}\left(f(a+h) - f(a)\right) = \lim_{h\to 0}\frac{\left(f(a+h) - f(a)\right)}{h}.h$$

$$= \lim_{h\to 0}\frac{\left(f(a+h) - f(a)\right)}{h}.\lim_{h\to 0} h = f'(a).0 = 0$$

{using (1)}

$\Rightarrow \lim_{h\to 0} f(a+h) = f(a) \Rightarrow$ f is continuous at x = a.

Further, if f is differentiable at every point of its domain, then it is continuous in that domain.

Example 1. Examine the differentiability of the function $f(x) = \begin{cases} \dfrac{\cos x - 1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ at x = 0 and comment on continuity at that point.

Solution We have

$$f'(0) = \lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0}\frac{\frac{\cos x - 1}{x} - 0}{x - 0}$$

$$= \lim_{x\to 0}\frac{\cos x - 1}{x^2} = \lim_{x\to 0}\frac{-\sin^2 x}{x^2(\cos x + 1)}$$

$$= -\left[\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\right]\left[\lim_{x\to 0}\frac{1}{\cos x + 1}\right] = -\frac{1}{2}.$$

Since, f is differentiable at x = 0, and using the above theorem, it is also continuous at x = 0.

Example 2. Given $f(x) = x^2 . \operatorname{sgn}(x)$, examine the continuity and derivability at x = 0.

Solution

$$f(x) = \begin{bmatrix} x^2 & \text{if} & x > 0 \\ 0 & \text{if} & x = 0 \\ -x^2 & \text{if} & x < 0 \end{bmatrix}$$

$$f'(0^+) = \lim_{h\to 0}\frac{h^2 - 0}{h} = 0$$

$$f'(0^-) = \lim_{h\to 0} -\frac{h^2 - 0}{-h} = 0$$

$\Rightarrow$ f is derivable at x = 0.

$\Rightarrow$ f is continuous at x = 0.

The converse of the above theorem is not necessarily true

If f is continuous at x = a, then f may or may not be differentiable at x = a.

For example, the functions $f(x) = |x|$ and

$$g(x) = x \sin\frac{1}{x} \text{ if } x \neq 0; \; g(0) = 0 \text{ are}$$

continuous at x = 0 but not differentiable at x = 0.

Since f is continuous at x = a, we have

$$\lim_{h\to 0}\left(f(a+h) - f(a)\right) = 0. \quad ...(1)$$

We need to check whether

$$f'(a) = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h} \text{ exists.} \quad ...(2)$$

Now (1) suggests that the limit (2) is in $\left(\frac{0}{0}\right)$ form, but we know that only some of these limits exist and hence we are not certain about the existence of the derivative. In fact, continuity of a function at a point is a necessary condition for its differentiability at this point. However, it should be noted that the continuity of a function at a point is not a sufficient condition for the derivative of this function to exist at that point, i.e. the continuity of a function at a point does not necessarily mean its differentiability at that point.

Thus, differentiability of a function is a stronger condition than continuity alone.

Example 3. Let $f(x) = \begin{cases} x^p \sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

Find the values of p for which (i) f is continuous but not differentiable at x = 0 (ii) f is continuous and differentiable at x = 0.

Solution f(0) = 0

For continuity, $\lim_{x\to 0} f(x) = 0$

$\therefore \quad \lim_{x\to 0} x^p \sin\frac{1}{x} = 0$.

This is possible only when $p > 0$. ...(1)

For differentiability,

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h^p \sin\frac{1}{h} - 0}{h}$$

should exist i.e. $\lim_{h\to 0} h^{p-1}\sin\frac{1}{h}$ should exist.

Thus, $f'(0)$ will exist only when $p > 1$

$\therefore$ f(x) will not be differentiable if $p \le 1$ ...(2)

From (1) and (2),

(i) f is continuous but not differentiable at $x = 0$ if $0 < p \le 1$.

(ii) f is continuous and differentiable at $x = 0$ if $p > 1$.

Example 4. If $f(x) = \begin{cases} ax + b \text{ for} & \text{for } x \le -1 \\ ax^3 + x + 2b & x > -1 \end{cases}$ is differentiable for at $x = -1$, find a and b.

Solution For differentiability, continuity is a necessary condition. Hence L.H.L. = R.H.L. at $x = -1$.

$\Rightarrow -a + 1 = -a - 1 + 2b$

$\Rightarrow -a - 1 + b + a = 0 \qquad \Rightarrow b = 1$.

Now, $f'(-1^-) = \lim_{h\to 0}\frac{f(-1-h)-f(-1)}{-h}$

$= \lim_{h\to 0}\frac{a(+1+h) - b + (b-a)}{+h}$

$= \lim_{h\to 0}\frac{a + ah - a}{h} = a$

and, $f'(-1^+) = \lim_{h\to 0}\frac{[a(-1+h)^3 + (-1+h) + 2b] - [b-a]}{h}$

$= \lim_{h\to 0}\frac{ah(h^2 - 3h + 3) + h + b - 1}{h}$ {using $b = 1$}

$= 3a + 1$.

Hence $f'(-1^+) = f'(-1^-) \Rightarrow 3a + 1 = a \Rightarrow a = -\frac{1}{2}$.

Thus, $a = -\frac{1}{2}$, $b = 1$.

Example 5. A function f is defined as,

$$f(x) = \begin{cases} \frac{1}{|x|} & \text{if } |x| \ge \frac{1}{2} \\ a + bx^2 & \text{if } |x| < \frac{1}{2} \end{cases}.$$

If f (x) is derivable at $x = 1/2$ find the values of a and b.

Solution $f(x) = \begin{cases} \frac{1}{|x|}, & x \le -\frac{1}{2},\ x \ge \frac{1}{2} \\ a + bx^2, & -\frac{1}{2} < x < \frac{1}{2} \end{cases}$

$$= \begin{cases} \frac{1}{-x}, & x \le -\frac{1}{2} \\ \frac{1}{x}, & x \ge \frac{1}{2} \\ a + bx^2, & -\frac{1}{2} < x < \frac{1}{2} \end{cases}$$

$$f'\left(\frac{1}{2}^+\right) = \lim_{h\to 0}\frac{f\left(\frac{1}{2}+h\right) - f\left(\frac{1}{2}\right)}{h} = \lim_{h\to 0}\frac{\frac{1}{\left(\frac{1}{2}+h\right)} - 2}{h}$$

$$= \lim_{h\to 0}\frac{1 - 2\left(\frac{1}{2}+h\right)}{h\left(\frac{1}{2}+h\right)} = \lim_{h\to 0}\frac{-2h}{h\left(\frac{1}{2}+h\right)} = -4.$$

$$f'\left(\frac{1}{2}^-\right) = \lim_{h\to 0}\frac{f\left(\frac{1}{2}-h\right) - f\left(\frac{1}{2}\right)}{-h}$$

$$= \lim_{h\to 0}\frac{a + b\left(\frac{1}{2}-h\right)^2 - 2}{-h} = \lim_{h\to 0}\frac{\left(a + \frac{b}{4} - 2\right) - bh + h^2}{-h}$$

For existence of this limit $a + \frac{b}{4} = 2$ ....(1)

$\Rightarrow$ f must be continuous at $x = \pm 1/2$.

Then, $f'\left(\frac{1}{2}^-\right) = b$.

Now $f'\left(\frac{1}{2}^-\right) = f'\left(\frac{1}{2}^+\right)$

$\Rightarrow \quad b = -4$.

Using (1), $a - 1 = 2 \Rightarrow a = 3$.

Hence, $a = 3$ and $b = -4$.

Theorem If a function f is discontinuous at $x = a$ then it is non-differentiable at $x = a$.

Proof If a function f is discontinuous at $x = a$ then the equation $\lim_{h\to 0}\left(f(a+h) - f(a)\right) = 0$ does not hold true.

For f to differentiable at $x = a$,

$f'(a)=\lim_{h\to0}\frac{f(a+h)-f(a)}{h}$ must exist.

We know that this limit can only exist in $\left(\frac{0}{0}\right)$ form, which is not possible because the numerator does not approach to 0 because the condition

$\lim_{h\to0}\left(f(a+h)-f(a)\right)=0$ does not hold true.

Hence, f is not differentiable at x = a.

Example 6. Comment on the differentiability of

$$f(x)=\begin{cases} x & , \quad x<1 \\ x+2 & , \quad x\ge1 \end{cases} \text{ at } x=1.$$

Solution L.H.D. $=f'(1^-)=\lim_{h\to0}\frac{f(1-h)-f(1)}{-h}$

$$=\lim_{h\to0}\frac{1-h-3}{-h}=\infty.$$

R.H.D. $=f'(1^+)=\lim_{h\to0}\frac{f(1+h)-f(1)}{h}$

$$=\lim_{h\to0}\frac{(1+h+2)-3}{h}=\lim_{h\to0}\frac{h}{h}=1.$$

Since L.H.D. ≠ R.H.D. f(x) is not differentiable at x = 1.

If we use the previous theorem, we see that

L.H.L. $=f(1^-)=1$ and R.H.L. $=f(1^+)=3$

f(1) = 3. Thus, the function is discontinuous and hence non-differentiable. We need not find L.H.D. and R.H.D. here to find differentiability.

Theorem If a function f is non-differentiable at x = a but both the one-sided derivatives exist (though being unequal), then f is continuous at x = a.

Proof To prove that f is continuous at x = a, we must prove that $\lim_{h\to0}\left(f(a-h)-f(a)\right)=0$ ...(1)

and $\lim_{h\to0}\left(f(a+h)-f(a)\right)=0$ ...(2)

Since L.H.D. exists at x = a,

$$f'(a^-)=\lim_{h\to0}\frac{f(a-h)-f(a)}{-h} \text{ exists.} \quad ...(3)$$

Now $\lim_{h\to0}\left(f(a-h)-f(a)\right)=\lim_{h\to0}\frac{\left(f(a-h)-f(a)\right)}{-h}.(-h)$

$$=\lim_{h\to0}\frac{\left(f(a-h)-f(a)\right)}{-h}.\lim_{h\to0}(-h) = f'(a^-).0 = 0.$$

{using (3)}

Since R.H.D. exists at x = a,

$$f'(a^+)=\lim_{h\to0}\frac{f(a+h)-f(a)}{h} \text{ exists.} \quad ...(4)$$

Now $\lim_{h\to0}\left(f(a+h)-f(a)\right)=\lim_{h\to0}\frac{\left(f(a+h)-f(a)\right)}{h}.h$

$$=\lim_{h\to0}\frac{\left(f(a+h)-f(a)\right)}{h}.\lim_{h\to0}h = f'(a).0 = 0$$

{using (4)}

Thus, both the conditions (1) and (2) are established. Hence, the function is continuous at x = a.

Note that if any of the one-sided derivatives at x = a does not exist then, f may or maynot be continuous at x = a.

STUDY TIP

Differentiable ⇒ Continuous

Continuous ⇏ Differentiable

Discontinuous ⇒ Non-differentiable

Both one-sided derivatives exist ⇒ Continuous

Non-differentiable ⇏ Discontinuous

Example 7. If $f(x)=\frac{xe^{1/x}}{1+e^{1/x}}$, $x\ne0$, $f(0)=0$, examine the continuity and differentiability at $x=0$.

Solution We check the differentiability first.

$$f'(0^-)=\lim_{x\to0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0-0}\frac{\frac{xe^{1/x}}{1+e^{1/x}}-0}{x}$$

$$=\lim_{x\to0-0}\frac{e^{1/x}}{1+e^{1/x}}=0.$$

$$f'(0^+)=\lim_{x\to0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0^+}\frac{\frac{xe^{1/x}}{1+e^{1/x}}-0}{x}$$

$$=\lim_{x\to0^+}\frac{e^{1/x}}{1+e^{1/x}}=\lim_{x\to0^+}\frac{1}{e^{-1/x}+1}=1.$$

Since $f'(0^-)\ne f'(0^+)$, f(x) is non-differentiable at x = 0.

To check continuity, we use the previous theorem.

Since both the one-sided derivatives exist (though they are unequal), f is continuous at x = 0.

We verify this result below :

We have $f(0^-)=\lim_{x\to0^-}\frac{xe^{1/x}}{1+e^{1/x}}=0$

$$f'(0^+) = \lim_{x\to 0^+} \frac{xe^{1/x}}{1+e^{1/x}} = \lim_{x\to 0^+} \frac{x}{1+e^{-1/x}} = 0$$

Also $f(0) = 0$

$\therefore$ $f(0^-) = f(0^+) = f(0) \Rightarrow f$ is continuous at $x = 0$.

Example 8. Show that the function $f : R \to R$ defined by $f(x) = x\left\{1+\frac{1}{3}\sin(\ln x^2)\right\}$, $x \neq 0$ and $f(0) = 0$ is everywhere continuous, but has no differential coefficient at the origin.

Solution Let us first check the differentiability

$$f'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{(0-h)\left\{1+\frac{1}{3}\sin(\ln h^2)\right\}-0}{(-h)}$$

$$= \lim_{h\to 0} \left\{1+\frac{1}{3}\sin(\ln h^2)\right\}$$

= does not exist.

Hence f(x) is not differentiable at $x = 0$. We cannot say anything about continuity immediately.

Now we check the continuity at $x = 0$:

$$\lim_{x\to 0} f(x) = \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0} \left\{1+\frac{1}{3}\sin(\ln h^2)\right\} = 0$$

Finally f(x) is continuous at $x = 0$ but not differentiable at $x = 0$.

Example 9. If $f(x) = \begin{cases} x^2 \operatorname{sgn}[x] + \{x\}, & 0 \le x < 2 \\ \sin x + |x-3|, & 2 \le x < 4 \end{cases}$, comment on the continuity and differentiability of f(x) at $x = 1, 2$.

Solution Looking at the kind of functions involved we check the continuity first.

Continuity at $x = 1$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2 \operatorname{sgn}[x] + \{x\})$$

$$= 1 \operatorname{sgn}(0) + 1 \quad = 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2 \operatorname{sgn}[x] + \{x\})$$

$$= 1 + 0 = 1$$

$f(1) = 1$.

Since, L.H.L = R.H.L = f(1), f(x) is continuous at $x = 1$.

Now, differentiability at $x = 1$,

$$f'(1^-) = \lim_{h\to 0^-} \frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0^+} \frac{(1-h)^2 \operatorname{sgn}[1-h] + 1 - h - 1}{-h} = 1$$

$$f'(1^+) = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0^+} \frac{(1+h)^2 \operatorname{sgn}[1+h] + \{1+h\} - 1}{h}$$

$$= \lim_{h\to 0^+} \frac{(1+h)^2 + h - 1}{h} = \lim_{h\to 0^+} \frac{h^2+3h}{h} = 3.$$

Here, $f'(1^+) \neq f'(1^-)$.

Hence f(x) is non-differentiable at $x = 1$.

Now at $x = 2$,

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 \operatorname{sgn}[x] + \{x\} = 4 . 0 + 1 = 1$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (\sin x + |x-3|) = 1 + \sin 2$$

Since, L.H.L ≠ R.H.L, f(x) is discontinuous at $x = 2$ and hence it is non-differentiable at $x = 2$.

Note: It can be proved that a function f is continuous from the left at those points where it is differentiable from the left, and f is continuous from the right at those points where it is differentiable from the right.

Now, consider the the function,

$$f(x) = 2x\sqrt{x^3-1} + 5\sqrt{x}\ \sqrt{1-x^4} + 7x^2\sqrt{x-1} + 3x$$

Here the domain of the function is $x \in \{1\}$

Since, it is a point function, there is no question of continuity and hence differentiability at $x = 1$.

$\therefore$ f(x) is neither continuous nor differentiable at $x = 1$.

Suppose that the function f is continuous everywhere. At how many points do you suspect that f can fail to be differentiable ? One might think that if a function is continuous on an interval, then it might fail to be derivable at finitely many points at the most. This, however, is far from the truth.

What ′s the worst such function you can think of ? Can a continuous function fail to have a derivative at every point? The answer, surprisingly enough, is yes. There exist functions which are continuous on R but which are not derivable at any point whatsoever. It did give a jolt to mathematicians when, Weierstrass gave an example of such a function.

Consider one of his functions

$$f(x)=\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^{n}\cos(9^{n}\pi x),$$

a formula that expresses f as an infinite sum of cosines with increasingly higher frequencies. By adding wiggles to wiggles infinitely many times, the formula produces a graph that is too bumpy in the limit to have a tangent anywhere. Another example of such a function defined on R is

$$f(x)=\sum_{n=1}^{\infty}\frac{1}{2^{n}}\cos(3^{n}x),\text{ for all } x\in R,$$

It can be shown that f is continuous everywhere but is not derivable anywhere. The proof of these assertions is, however, beyond the scope of the present book.

Remark In the definition of derivative we are regarding the derivative $f'(x_0)$ as a real number associated with the difference quotient $(f(x) - f(x_0))/(x - x_0)$ considered as a function which has values for each x in the domain of f with the exception of x_0. In fact, the definition merely equates the ideas of derivability of f at x_0 with the notion of continuous extension of the difference quotient to x_0.

For if we write $Q(x)=(f(x)-f(x_0))/(x-x_0)$, the domain of Q(x) is $D_f-\{x_0\}$ so that x_0 is a removable discontinuity for Q(x), and we have the continuous extension Q*(x) defined to be Q(x) for $x\neq x_0$ with $Q^*(x_0)=f'(x_0)$. In symbols,

$$Q^*(x)=\begin{cases}\dfrac{f(x)-f(x_0)}{x-x_0}, & x\in D_f-\{x_0\}\\ f'(x_0), & x=x_0\end{cases}$$

Thus, for instance, if f is continuous on its domain, then Q(x) is continuous on the same domain minus x_0, but the continuous extension Q*(x) would be continuous on the whole domain of f. The latter idea is expressed geometrically by rotating secant line through a given point on a curve continuously to a tangent line as a limiting line.

Theorem Suppose f is differentiable at x = c. Then $f(x)=f(c)+f'(c)(x-c)+e(x)(x-c)$,

where $e(x)\to 0$ as $x\to c$ and $e(c)=0$.

Conversely, let f be defined on an interval D and let $c \in D$. Suppose there is a constant K such that $f(x) = f(c) + K(x - c) + e(x)(x - c)$, where $e(x) \to 0$ as $x \to c$. Then f is differentiable at c and $f'(c) = K$.

Proof If $\lim\limits_{x\to c}\dfrac{f(x)-f(x)}{x-c}=f'(c)$ exists, then

$$e(x)=\frac{f(x)-f(c)}{x-c}-f'(c)\to 0\quad\text{as } x\to c$$

Conversely, given $f(x) = f(c) + K(x - c) + e(x)(x - c)$, where $e(x) \to 0$, then

$$\frac{f(x)-f(c)}{x-c}=K+e(x)\to K$$

Hence the derivative exists and equals K.

Example 10. Check the continuity and differentiability of f(x) at x = 1, where $f(x)=|x-1|([x]-\{x\})$.

Solution $$f'(1^+)=\lim_{h\to0}\frac{f(1+h)-f(1)}{h}$$

$$=\lim_{h\to0}\frac{|h|([1+h]-\{1+h\})-0}{h}=\lim_{h\to0}\frac{h(1-h)}{h}=1.$$

$$f'(1^-)=\lim_{h\to0}\frac{f(1)-f(1-h)}{h}$$

$$=\lim_{h\to0}\frac{0-|-h|([1-h]-\{1-h\})}{h}$$

$$=\lim_{h\to0}\frac{-h[0-(1-h)]}{h}=-1$$

Since $f'(1^+)$ and $f'(1^-)$ are not equal, hence f is not differentiable at x = 1. Since, the one-sided derivatives exist, f is continuous at x = 1.

Example 11. Let
$$f(x)=\begin{cases}e^{\frac{x^2+1}{x}}, & x<0\\ 0, & x=0\\ x^2, & x>0\end{cases}$$

Discuss continuity and differentiability of f(x) at x = 0.

Solution At x = 0

$$f'(0^-)=\lim_{h\to0}\frac{f(0-h)-f(0)}{-h}$$

$$=\lim_{h\to0}\frac{e^{-\left(h+\frac{1}{h}\right)}-0}{-h}=\lim_{h\to0}e^{-h}\cdot\frac{e^{-\frac{1}{h}}}{-h}$$

$$=\lim_{h\to0}\frac{e^{-\frac{1}{h}}}{-h}\quad\text{put } t=-\frac{1}{h}\text{ we get}$$

$$=\lim_{t\to-\infty}te^{t}=\lim_{t\to-\infty}\frac{t}{e^{-t}}=\lim_{t\to-\infty}\frac{t}{-e^{-t}}$$

(applying L' Hospital rule)

$= 0.$

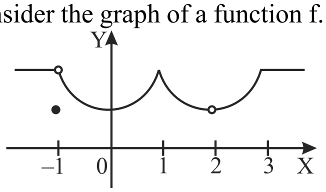
$$f'(0^+)=\lim_{h\to0}\frac{f(h)-f(0)}{h}=\lim_{h\to0}\frac{h^2-0}{h}$$

$$=\lim_{h\to0}h=0.$$

Hence, function is continuous and differentiable at x = 0.

Concept Problems B

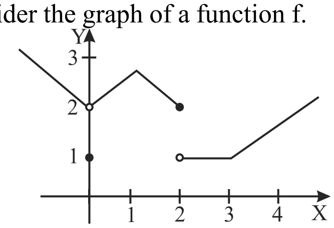
1. Consider the graph of a function f.



(a) For which numbers a does $\lim_{x \to a} f(x)$ exist, but f is not continuous at a ?

(b) For which numbers a is f continuous at a, but not differentiable at a ?

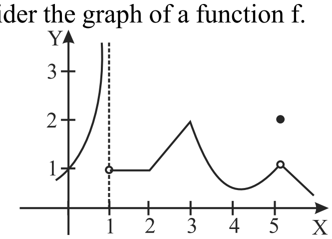
2. Consider the graph of a function f.



(a) For which numbers a does $\lim_{x \to a} f(x)$ exist, but f is not continuous at a ?

(b) For which numbers a is f continuous at a, but not differentiable at a ?

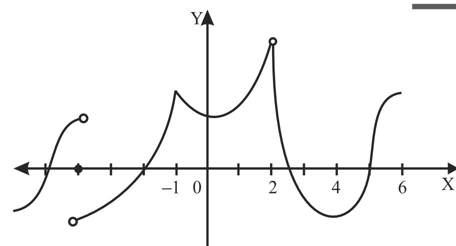
3. Consider the graph of a function f.



(a) For which numbers a does $\lim_{x \to a} f(x)$ exist, but f is not continuous at a ?

(b) For which numbers a is f continuous at a, but not differentiable at a ?

4. The graph of f is shown. State, with reasons, the number at which f is not differentiable.



5. Let $f(x) = \begin{cases} 2 - x & \text{if } x \le 1 \\ x^2 - 2x + 2 & \text{if } x > 1 \end{cases}$ Is f differentiable at 1? Sketch the graphs of f and f′.

6. Let $f(x) = \begin{cases} -2x & \text{if } x < 1 \\ \sqrt{x} - 3 & \text{if } x \ge 1 \end{cases}$. Show that f is continuous but not differentiable at x = 1.

7. If $f(x) = \begin{cases} ax^2 + 1, & x \le 1 \\ x^2 + ax + b, & x > 1 \end{cases}$ is differentiable at x = 1, then find a and b.

8. If $f(x) = \begin{cases} \frac{1}{x} \sin x^2, & x \ne 0 \\ 0, & x = 0 \end{cases}$, discuss the continuity and differentiability of f(x) at x = 0.

9. A function f is defined as follows :

$$f(x) = \begin{cases} 1 & \text{for } -\infty < x < 0 \\ 1 + \sin x & \text{for } 0 \le x < \frac{\pi}{2} \\ 2 + \left(x - \frac{\pi}{2}\right)^2 & \text{for } \frac{\pi}{2} \le x < +\infty \end{cases}.$$

Discuss the continuity and differentiability at x = 0 and $x = \pi/2$.

10. If $f(x) = \begin{cases} e^{x^2 + x}, & x > 0 \\ ax + b, & x \le 0 \end{cases}$ is differentiable at x = 0, then find a and b.

11. If $f(x) = \begin{cases} [2x] + x, & x < 1 \\ \{x\} + 1, & x \ge 1 \end{cases}$ comment on the continuity and differentiable at x = 1.

Practice Problems B

12. Check the continuity and differentiability of

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \le x \le 1 \\ x^3 - x + 1, & x > 1 \end{cases}$$ at x = 0 and x = 1.

13. Test the continuity and differentiability of the function

$$f(x) = \begin{cases} \dfrac{x(2^{1/x} - 2^{-1/x})}{(2^{1/x} + 2^{-1/x})}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

at x = 0.

14. Find the value of 'a' for which

$$f(x)=\begin{cases}\operatorname{sgn}(\cos x) & , \quad x<\pi/2 \\ a-\sin x & , \quad x\geq \pi/2\end{cases}$$ is differentiable at $x=\pi/2$.

15. Check the differentiability of the function $f(x)=\sin^{-1}\left(\dfrac{2\cos x}{1+\cos^2 x}\right)$ at $x=0, \pi$.

16. Examine for continuity and differentiability at the points $x = 1$ and $x = 2$, the function f defined by

$$f(x)=\begin{cases}x[x] & , \quad 0\leq x<2 \\ (x-1)\,[x] & , \quad 2\leq x\leq 3\end{cases}$$

where $[x]$ = greatest integer less than or equal to x.

17. Show that the function

$$f(x)=\begin{cases}\sin x.\cos(1/x) & \text{at} \quad x\neq 0, \\ 0 & \text{at} \quad x=0.\end{cases}$$

is continuous at the point $x = 0$, but does not have even one sided derivatives.

18. If $f(x)=\begin{cases}\sin\dfrac{\pi x}{2}, & x<1 \\ [2x-3]x, & x\geq 1\end{cases}$, where [.] denotes the greatest integer function, then find whether it is differentiable at $x = 1$.

3.5 DERIVABILITY AT ENDPOINTS

Till now, the derivative $f'(x)=\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$ was calculated when the function f was defined in a neighbourhood of the point x, where Δx tends to zero attaining both positive and negative values.

Suppose that a function is defined only in the left neighbourhood of x, then the above limit can be calculated when Δx tends to zero running through negative values ($\Delta x < 0$) only. This is termed as the left hand derivative of f at x (denoted as $f'(x^-)$). If this exists, then it is considered as the derivative of the function at x.

Similarly, if a function is defined only in the right neighbourhood of x, then the above limit can be calculated when Δx tends to zero running through positive values ($\Delta x > 0$) only. This is termed as the right hand derivative of f at x (denoted as $f'(x^+)$). If this exists, then it is considered as the derivative of the function at x.

Definition If a function f is defined only in the left neighbourhood of a point $x = a$ (for $x \leq a$), it is said to be differentiable at a if

$$\text{L.H.D.} = f'(a^-)=\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h} \text{ exists.}$$

In such a case $f'(a) = f'(a^-)$.

Similarly, if a function f is defined only in the right neighbourhood of a point $x = a$ (for $x \geq a$), it is said to be differentiable at a if

$$\text{R.H.D.} = f'(a^+)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \text{ exists.}$$

In such a case $f'(a) = f'(a^+)$.

For example, $f(x) = x^{3/2}$ is differentiable at $x = 0$ but $g(x)=\sqrt{x}$ is not differentiable at $x = 0$.

Since, f and g are defined only in the right neighbourhood of a point $x = 0$, we calculate only R.H.D.

$$f'(0^+)=\lim_{h\to 0}\frac{h^{3/2}-0}{h} = \lim_{h\to 0}h^{1/2} = 0 \text{ exists.}$$

Hence $f'(0) = 0$.

$$g'(0^+)=\lim_{h\to 0}\frac{h^{1/2}-0}{h} = \lim_{h\to 0}h^{-1/2} = \infty$$

does not exist.

Hence, $\sqrt{x}$ is not differentiable at $x = 0$.

Note: The four inverse trigonometric functions $\sin^{-1}x$, $\cos^{-1}x$, $\operatorname{cosec}^{-1}x$, and $\sec^{-1}x$ are not differentiable at the points $x = \pm 1$.

Consider $f(x) = \sin^{-1}x$.

At $x = 1$, f is defined only in the left neighbourhood.

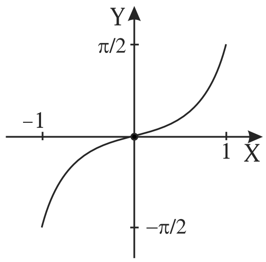
$$\text{Hence, } f'(1^-)=\lim_{h\to 0}\frac{f(1-h)-f(0)}{-h}$$

$$=\lim_{h\to 0}\frac{\sin^{-1}(1-h)-\sin^{-1}1}{-h}$$

$$=\lim_{h\to 0}\frac{\sin^{-1}(1-h)-\pi/2}{-h}=\lim_{h\to 0}\frac{-\cos^{-1}(1-h)}{-h}$$

$$=\lim_{h\to 0}\frac{\sin^{-1}\sqrt{2h-h^2}}{h}=\infty$$

Thus, $f(x) = \sin^{-1}x$ is not differentiable at $x = 1$.



There is a vertical tangent to the graph of f at x = 1.

Example 1. Discuss the differentiability of the function

$f(x) = x\left(\sqrt{x} - \sqrt{(x+1)}\right)$ at $x = 0$.

Solution Since domain of f(x) is $[0, \infty)$, we find the R.H.D.

$$\therefore \quad f'(0^+) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{h(\sqrt{h}-\sqrt{h+1})-0}{h}$$

$$= \lim_{h\to 0}(\sqrt{h}-\sqrt{h+1}) = -1$$

Hence, f is differentiable at x = 0.

3.6 DIFFERENTIABILITY OVER AN INTERVAL

f (x) is said to be differentiable in an interval if it is differentiable at each and every point of the interval.

Differentiability over an open interval

A function f(x) is said to be differentiable in an open interval (a, b) if it is differentiable at all interior points in (a, b).

For example, $f(x) = |x-1|$ is differentiable in (0, 1).

$y = \sin^{-1}x$ is differentiable in (–1, 1).

Similarly, a function f(x) is said to be differentiable in the open interval (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$ if it is differentiable at all interior points in the interval.

For example, $f(x) = \sqrt{x}$ is differentiable in $(0, \infty)$.

Differentiability over a closed interval

A function f(x) is said to be differentiable in a closed interval [a, b] if

(i) it is differentiable at all interior points in (a, b).

(ii) R.H.D. = $f'(a^+)$ exists at the left endpoint a.

(ii) L.H.D. = $f'(a^-)$ exists at the right endpoint b.

If a function is defined in a closed interval then, we by derivatives at the end points of the interval, we always mean the one-sided derivatives : the right hand derivative at the left end point and the left hand derivative at the right end point. This interpretation of differentiability at the end points of the interval of definition is analogous to that of continuity.

For example, $f(x) = x^{3/2}$ is differentiable in [0, 1].

$f(x) = \sin x$ is differentiable in $[0, 2\pi]$.

It should be noted that not all the elementary functions are differentiable everywhere in their domain.

Note:

1. All polynomial, exponential, logarithmic and trigonometric functions (inverse trigonometric not included) are differentiable at each point in their domain.
2. Modulus function and signum function are non differentiable at x = 0. Hence, $y = |f(x)|$ and $y = \text{sgn}(f(x))$ should be checked at points where $f(x) = 0$.
3. The inverse trigonometric functions $y = \sin^{-1}x$, $\cos^{-1}x$, $\text{cosec}^{-1}x$, and $\sec^{-1}x$ are not differentiable at the points $x = \pm 1$. Hence, $y = \sin^{-1}(f(x)), \cos^{-1}(f(x)), \text{cosec}^{-1}(f(x))$, and $\sec^{-1}(f(x))$ should be checked at points where $f(x) = \pm 1$.
4. Greatest integer function and fractional part functions are non differentiable at all integral x. Hence, $y = [f(x)]$ and $y = \{f(x)\}$ should be checked at points where $f(x) = n,\ n \in I$.
5. Further, a function should be checked at all those points where discontinuity may arise.

Example 1. Show that $f(x) = \dfrac{x}{1+|x|}$ is differentiable for all $x \in R$.

Solution Since $f(x) = \begin{cases} \dfrac{x}{1+x}, & x > 0 \\ 0, & x = 0, \\ \dfrac{x}{1-x}, & x < 0 \end{cases}$

clearly f is differentiable for all $x \in R - \{0\}$. We need to check at x = 0.

$$f'(0^+) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{\frac{h}{1+h}-0}{h}$$

$$= \lim_{h\to 0}\frac{1}{1+h} = 1$$

$$f'(0^-) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{\frac{-h}{1+h}-0}{-h}$$

$$= \lim_{h\to 0}\frac{1}{1+h} = 1$$

$\Rightarrow$ $f'(0^+) = f'(0^-)$. Hence f is differentiable at x = 0.

Finally, f is differentiable for all $x \in R$.

Example 2.

$$\text{Let } f(x) = \begin{cases} \left\{x+\frac{1}{3}\right\}[\sin \pi x] & 0 \le x < 2 \\ [2x]\text{sgn}\left(x-\frac{4}{3}\right) & 1 \le x \le 2 \end{cases}$$

where [.] and {.} denote the greatest integer function and fractional part function respectively. Find the points at which continuity and differentiability should be checked. Also check the continuity and differentiability of f(x) at x = 1.

Solution $f(x) = \begin{cases} \left\{x+\frac{1}{3}\right\}[\sin \pi x] & 0 \le x < 2 \\ [2x]\text{sgn}\left(x-\frac{4}{3}\right) & 1 \le x \le 2 \end{cases}$

The suspicious points for continuity and differentiability are obtained when :

(i) $x+\frac{1}{3}$ becomes an integer for $0 \le x < 2$

$\Rightarrow$ $x = \frac{2}{3}, \frac{4}{3}$.

(ii) $\sin \pi x$ becomes an integer for $0 \le x < 2$

$\Rightarrow$ $x = 0, \frac{1}{2}, 1, \frac{3}{2}$.

(iii) The point of change in definition is x = 1.

(iv) 2x becomes an integer for $1 \le x \le 2$

$\Rightarrow$ $x = 1, \frac{3}{2}, 2$.

(v) $x-\frac{4}{3}$ becomes zero for $1 \le x \le 2 \Rightarrow x = \frac{4}{3}$.

(vi) The end points of closed domain $[0, 2] \Rightarrow x = 0, 2$.
Hence, the points where we should check the continuity and differentiability are

$$x = 0, \frac{1}{2}, \frac{2}{3}, 1, \frac{4}{3}, \frac{3}{2}, 2.$$

We check continuity at x = 1.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}\left\{x+\frac{1}{3}\right\}[\sin \pi x] = 0$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}[2x]\,\text{sgn}\left\{x+\frac{1}{3}\right\} = 2\{-1\} = -2$$

Since, L.H.L $\neq$ R.H.L. f(x) is discontinuous at x = 1 and hence it is non-differentiable at x = 1.

Example 3. Check differentiability of f(x) in [–2, 2], if

$$f(x) = \begin{cases} \cos\frac{\pi}{2}(|x|-\{x\}) & x < 1 \\ \sqrt{4x^2-12x+9}\{x\} & x \ge 1 \end{cases}$$

where {.} denotes the fractional part function.

Solution We have

$$f(x) = \begin{cases} \cos\frac{\pi}{2}(|x|-\{x\}) & x < 1 \\ \sqrt{4x^2-12x+9}\{x\} & x \ge 1 \end{cases}$$

i.e. $f(x) = \begin{cases} \cos\frac{\pi}{2}(|x|-\{x\}) & x < 1 \\ |2x-3|\{x\} & x \ge 1 \end{cases}$

Now, we have

$|x| - \{x\} = |x| - x + [x] = -2x - 2, \quad -2 \le x < -1$
$= -2x - 1, \quad -1 \le x < 0$
$= 0, \quad 0 \le x < 1$

and $|2x-3|\{x\} = |2x-3|(x-[x])$
$= (3-2x)(x-1), \quad 1 \le x < 3/2$
$= (2x-3)(x-1), \quad 3/2 \le x < 2$
$= 0, \quad x = 2$

Thus, we have

$$f(x) = \begin{cases} \cos\frac{\pi}{2}(2x+2), & -2 \le x < -1 \\ \cos\frac{\pi}{2}(2x+1), & -1 \le x < 0 \\ \cos 0, & 0 \le x < 1 \\ (3-2x)(x-1), & 1 \le x < 3/2 \\ (2x-3)(x-1), & 3/2 \le x < 2 \\ 0, & x = 2 \end{cases}$$

We need to check differentiability of f(x) at
x = –2, –1, 0, 1, 3/2, 2. We first check continuity.

At x = 2, we have
$f(2) = 0$ and $f(2^-) = (2 . 2 - 3)(2 - 1) = 1$.

⇒ discontinuity at x = 2.

At x = 3/2, we have

$$f\left(\frac{3^+}{2}\right) = f\left(\frac{3}{2}\right) = 0$$

and $f\left(\frac{3^-}{2}\right) = \left(3 - 2.\frac{3}{2}\right)\left(\frac{3}{2} - 1\right) = 0$

⇒ continuity at x = 3/2.

At x = 1, we have $f(1^+) = f(1) = 0$ and $f(1^-) = \cos 0 = 1$

⇒ discontinuity at x = 1.

At x = 0, we have

$$f(0^+) = f(0) = \cos 0 = 1 \text{ and } f(0^-) = \cos\frac{\pi}{2} = 0$$

⇒ discontinuity at x = 0.

At x = –1, we have

$$f(-1^+) = f(-1) = \cos\frac{\pi}{2} = 0 \text{ and } f(-1^-) = \cos 0 = 1$$

⇒ discontinuity at x = –1.

At x = –2, we have $f(-2^+) = f(-2)$

⇒ continuity at x = –2.

Hence, we need to check differentiability only at x = –2, 3/2.

At x = –2 which is the left endpoint, we have

$$f'(-2^+) = \lim_{h\to 0}\frac{f(-2+h) - f(-2)}{h}$$

$$= \lim_{h\to 0}\frac{\cos\frac{\pi}{2}(-4+2h+2) - \cos\frac{\pi}{2}(-4+2)}{h}$$

$$= \lim_{h\to 0}\frac{\cos\pi(1-h)+1}{h} = \lim_{h\to 0}\frac{\pi\sin(\pi - \pi h)}{1} = 0$$

⇒ f is differentiable at x = –2.

At x = 3/2, we have

$$f'\left(\frac{3^+}{2}\right) = \lim_{h\to 0}\frac{f\left(\frac{3}{2}+h\right) - f\left(\frac{3}{2}\right)}{h}$$

$$= \lim_{h\to 0}\frac{2h\left(\frac{3}{2}+h-1\right)-1}{h} = 1$$

$$f'\left(\frac{3^-}{2}\right) = \lim_{h\to 0}\frac{f\left(\frac{3}{2}\right) - f\left(\frac{3}{2}-h\right)}{h}$$

$$= \lim_{h\to 0}\frac{0 - 2h\left(\frac{3}{2}-h-1\right)}{h} = -1$$

⇒ non-differentiable at x = 3/2

Hence, f is differentiable on [–2,2] except at x = –1, 0, 1, 3/2, 2.

Example 4. If $x + 4|y| = 6y$, then find whether y as a function of x is continuous and derivable.

Solution We have $x + 4|y| = 6y$

$$\Rightarrow y = f(x) = \begin{cases} \frac{1}{2}x, & x \ge 0 \\ \frac{1}{10}x, & x < 0 \end{cases}$$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0}\frac{1}{2}h = 0$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0}\frac{1}{10}(0-h) = 0$$

⇒ f is continuous at x = 0 and hence for all x ∈ R.

$$f'(0^+) = \lim_{h\to 0}\frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{h}{2} - 0}{h} = \frac{1}{2}$$

$$f'(0^-) = \lim_{h\to 0}\frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h\to 0}\frac{-\frac{h}{10} - 0}{-h} = \frac{1}{10}$$

∴ f is not derivable at x = 0.

⇒ f is derivable at all x except x = 0.

Example 5.

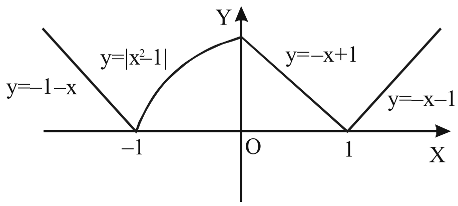
$$\text{If } f(x) = \begin{cases} -1-x, & x \le -1 \\ |x^2-1|, & -1 < x \le 0 \\ k(-x+1), & 0 < x \le 1 \\ |x-1|, & x > 1 \end{cases},$$

then find the value of k so that f(x) becomes continuous at x = 0. Hence, find all the points where the functions is non-differentiable.

Solution The only possible value of k is 1 which is obtained by equating the one-sided limits at x = 0.

Now, we draw the graph of y = f(x).

From the graph, it is clear that the function is non-differentiable at x = 0, ±1 because at each of these points the graph is having a corner.

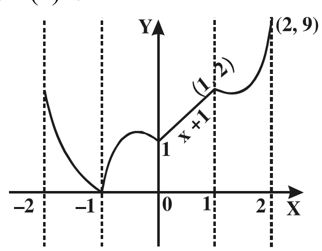


Example 6. If $f(x) = |x+1|\,(|x| + |x-1|)$, then draw the graph of $f(x)$ in the interval $[-2, 2]$ and discuss the continuity and differentiability in $[-2, 2]$

Solution Here, $f(x) = |x+1|\,(\,|x| + |x-1|\,)$

$$= \begin{cases} (x+1)(2x-1), & -2 \le x < -1 \\ -(x+1)(2x-1), & -1 \le x < 0 \\ (x+1), & 0 \le x < 1 \\ (x+1)(2x-1), & 1 \le x \le 2 \end{cases}$$

The graph of $f(x)$ is

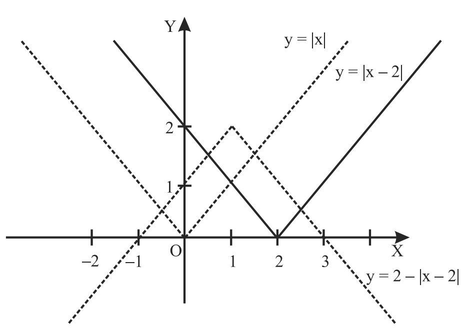


which is clearly, continuous for $x \in R$.
The graph has corners at $x = -1, 0, 1$.
Hence, $f(x)$ is differentiable for $x \in R - \{-1, 0, 1\}$.

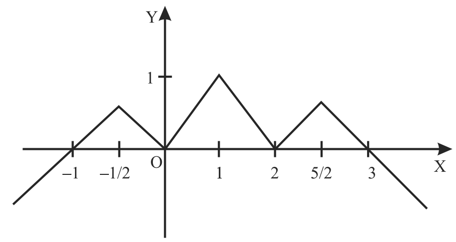
Example 7. If $f(x) = \text{Min}\,\{|x|, |x-2|, 2-|x-1|\}$, then draw the graph of $f(x)$ and also discuss its continuity and differentiability.

Solution $f(x) = \text{Min}\,\{|x|, |x-2|, 2-|x-1|\}$

We compare the graphs of
$y = |x|$, $y = |x-2|$ and $y = 2-|x-1|$ and choose the least value at each x.



Graph of $y = \text{Min}\,\{|x|, |x-2|, 2-|x-1|$:

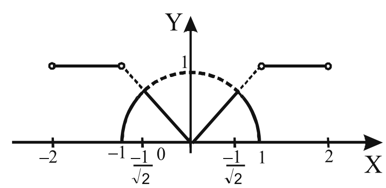


It is clear from the figure that $f(x)$ is continuous $\forall\, x \in R$ and non-differentiable at $x = -\dfrac{1}{2}, 0, 1, 2, \dfrac{5}{2}$.

Example 8. Consider the function

$$f(x) = \min(|x|, \sqrt{1-x^2}\,), -1 \le x \le 1$$
$$= [|x|], \qquad -1 < |x| < 2$$

Plot the curve and discuss its continuity and differentiability.



Solution $y = [|x|],\ x \in (-2, -1) \cup (1, 2)$

$\Rightarrow$ $y = 1$.

The curve shown by thick lines represents the curve.
From the graph, we can see that $f(x)$ is continuous on $x \in (-2, 2) - \{\pm 1\}$ and

differentiable on $x \in (-2, 2) - \left\{\pm 1, \dfrac{\pm 1}{\sqrt{2}}, 0\right\}$.

Example 9. Let $f(x) = x^3 - x^2 + x + 1$ and

$$g(x) = \begin{cases} \max\{f(t);\ 0 \le t \le x\}, & 0 \le x \le 1 \\ 3 - x; & 1 < x \le 2 \end{cases}.$$

Discuss the continuity and differentiability of the function $g(x)$ in the interval $(0, 2)$.

Solution $f(t) = t^3 - t^2 + t + 1$

$\therefore\ f'(t) = 3t^2 - 2t + 1$

Its discriminant $= (-2)^2 - 4.3.1 = -8 < 0$
and coefficient of $t^2 = 3 > 0$

Hence $f'(t) > 0$ for all real t.

$\Rightarrow$ $f(t)$ is strictly increasing

Thus $f(t)$ is maximum when t is maximum and $t_{max} = x$

$\therefore$ $\max f(t) = f(x)$

$$\Rightarrow \quad g(x) = \begin{cases} x^3 - x^2 + x + 1, & 0 \le x \le 1 \\ 3 - x, & 1 < x \le 2 \end{cases}$$

Now it can be easily seen that f(x) is continuous in (0, 2) and differentiable in (0, 2) except at x = 1 because at x = 1. LHD > 0 while RHD = – 1 < 0.

Example 10. Check the differentiability of the function $f(x) = \max\{\sin^{-1}|\sin x|, \cos^{-1}|\sin x|\}$.

Solution $\sin^{-1}|\sin x|$ is periodic with period π

$$\Rightarrow \quad \sin^{-1}|\sin x| = \begin{cases} x & , n\pi \le x \le n\pi + \dfrac{\pi}{2} \\ \pi - x, & n\pi + \dfrac{\pi}{2} \le x \le n\pi + \pi \end{cases}$$

Also $\cos^{-1}|\sin x| = \dfrac{\pi}{2} - \sin^{-1}|\sin x|$

$$\Rightarrow \quad f(x) = \max\begin{cases} \left\{x, \dfrac{\pi}{2} - x\right\} & , \quad n\pi \le x \le n\pi + \dfrac{\pi}{2} \\ \left\{\pi - x, x - \dfrac{\pi}{2}\right\} & , \quad n\pi + \dfrac{\pi}{2} \le x \le n\pi + \pi \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} \dfrac{\pi}{2} - x, & n\pi \le x \le n\pi + \dfrac{\pi}{4} \\ x, & n\pi + \dfrac{\pi}{4} < n\pi + \dfrac{\pi}{2} \\ \pi - x, & n\pi + \dfrac{\pi}{2} < x \le n\pi + \dfrac{3\pi}{4} \\ x - \dfrac{\pi}{2}, & n\pi + \dfrac{3\pi}{4} < x \le n\pi + \pi \end{cases}$$

$\Rightarrow$ f(x) is not differentiable at $x = 0, \dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{3\pi}{4}, \pi ...$

$\Rightarrow$ f(x) is not differentiable at $x = \dfrac{n\pi}{4}$.

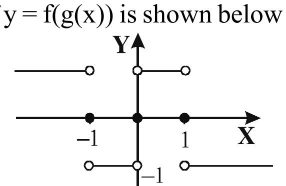
Example 11. Let f(x) = sgn x and $g(x) = x(1 - x^2)$. Investigate the composite functions f(g(x)) and g(f(x)) for continuity and differentiability.

Solution Since $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$

$$f(g(x)) = \begin{cases} 1 & \text{if } g(x) > 0 \\ 0 & \text{if } g(x) = 0 \\ -1 & \text{if } g(x) < 0 \end{cases}$$

$$= \begin{cases} 1 & \text{if } x \in (-\infty, -1) \cup (0, 1) \\ 0 & \text{if } x = 0, 1, -1 \\ -1 & \text{if } x \in (-1, 0) \cup (1, \infty) \end{cases}$$

The graph of y = f(g(x)) is shown below



From the graph we can easily conclude that f(g(x)) is discontinuous and non derivable at – 1, 0, 1.

Now, $g(f(x)) = f(x)(1 - f^2(x))$

$= \text{sgn}\, x\,(1 - \text{sgn}^2 x) = 0$ for all x,

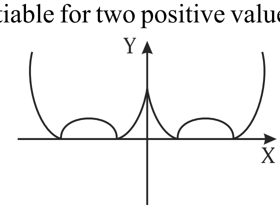
Hence g (f(x)) is continuous and derivable for all x.

Example 12. Find the interval of values of k for which the function $f(x) = |x^2 + (k-1)|x| - k|$ is non differentiable at five points.

Solution

$f(x) = |x^2 + (k-1)|x| - k| = |(|x| - 1)(|x| + k)|$

Also f(x) is an even function and f(x) is not differentiable at five points. So |(x – 1) (x + k)| is non differentiable for two positive values of x.



$\Rightarrow$ Both the roots of (x – 1) (x + k) = 0 are positive.

$\Rightarrow$ k < 0 $\Rightarrow$ k ∈ (–∞, 0).

Concept Problems **C**

1. Indicate the points at which the following functions have no finite derivative :

   (a) $y = \cos^{-1}\dfrac{x+1}{2}$ (b) $y = \sin^{-1}\dfrac{1}{x}$

2. Show that for any positive integer n, the root function $f(x) = x^{1/n}$ is not derivable at the origin. Is it continuous at the origin ?

3. (a) Find $f'(0^-)$ and $f'(0^+)$ given f(x) = |x|.

(b) The function $f(x) = |3x - 10|$ is differentiable except at a single point. What is this point, and what are the values of its left hand and right hand derivatives of f there ?

4. Sketch the graph of the function $f(x) = x + |x|$. Then investigate its differentiability. Find the derivative $f'(x)$ where it exists. Also find the one-sided derivatives at the points where $f'(x)$ does not exist.

5. Investigate the differentiability of the functions:

(i) $f(x) = |x^3|$

(ii) $f(x) = \begin{cases} \dfrac{12}{(5-x)^2} & \text{if } x < 3, \\ x^2 - 3x + 3 & \text{if } x \geq 3 \end{cases}$

6. Discuss the continuity and differentiability of the function $f(x) = \sin x + \sin|x|, x \in R$. Draw a rough sketch of the graph of $f(x)$.

7. (a) Sketch the graph of the function $f(x) = x|x|$.

(b) For what value of x is f differentiable?

(c) Find a formula for $f'(x)$.

8. If $f(x) = [x] + [1 - x] = -1 \leq x \leq 3$, Draw its graph and comment on the continuity and differentiability of f(x).

9. Show that the Dirichlet function is nowhere derivable.

10. Examine the differentiability of

$$f(x) = \begin{cases} xe^{|x|} & , x < 1 \\ ex^2 & , x \geq 1 \end{cases}.$$

Practice Problems C

11. Let $f(x) = \dfrac{\sqrt{x+1} - 1}{\sqrt{x}}$ for $x \neq 0$, $f(0) = 0$. Is the function f(x) continuous and differentiable at $x = 0$?

12. Investigate the differentiability of the function

$$f(x) = \begin{cases} 11 + 6x - x^2 & \text{if } x < 3 \\ x^2 - 6x + 29 & \text{if } x \geq 3 \end{cases}.$$

13. If $f(x) = \begin{cases} \min\{x, x^2\}, x \geq 0 \\ \max\{2x, x^2 - 3\}, x < 0 \end{cases}$ then find the points of non differentiability of f(x).

14. Find a and b, given that the derivative of f(x) exists everywhere, when

$$f(x) = \begin{cases} ax^2 - bx + 2, & x < 3 \\ bx^2 - 3, & x \geq 3 \end{cases}.$$

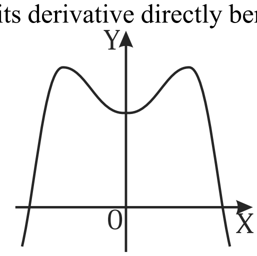
15. Check differentiability of f(x) at $x = 0$,

where $f(x) = x\left[1 + \frac{1}{3}\sin(\ln x^2)\right], \quad x \neq 0$

$= 0, \quad x = 0$

where [.] represents the greatest integer.

16. Copy the graph of the function. Then sketch a graph of its derivative directly beneath.



17. $f(x) = \begin{cases} 1 - x & , (0 \leq x \leq 1) \\ x + 2 & , (1 < x < 2) \\ 4 - x & , (2 \leq x \leq 4) \end{cases}$.

Discuss the continuity and differentiability of $y = f(f(x))$ for $0 \leq x \leq 4$.

18. Draw the graph of $f(x) = \max.\{\sin x, \cos x, 1 - \cos x\}$ and find the number of points belonging to $(0, 2\pi)$ where f(x) is non differentiable.

19. Let $f(x) = [x]$ and $g(x) = \begin{cases} 0 \text{ if } x \text{ is an integer} \\ x^2 \text{ otherwise} \end{cases}$ then define g o f and f o g and examine their differentiability.

20. Let $f(x) = \cos x$ and

$$g(x) = \begin{cases} \text{minimum } \{f(t) ; 0 \leq t \leq x\}, & x \in [0, \pi] \\ \sin x - 1, & x > \pi \end{cases}.$$

Draw the graph of g(x) and comment on differentiability.

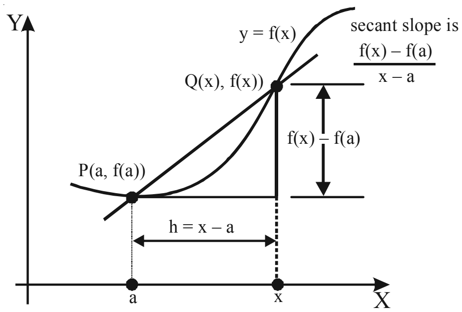
3.7 ALTERNATIVE LIMIT FORM OF THE DERIVATIVE

The following is an alternative limit form of the derivative. The derivative of f at $x = a$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to c} \frac{f(x) - f(a)}{x - a}$$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

The use of this formula simplifies some derivative calculations.



The one-sided derivatives of f at a are given by

$$f'(a^+) = \lim_{x\to a^-}\frac{f(x)-f(a)}{x-a} \text{ and } f'(a^-) = \lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}.$$

$f'(a)$ exists if they are equal.

Example 1. If f is a differentiable function with $f'(2) = 3$ then find the value of $\lim_{x\to 2}\frac{f(x)-f(2)}{\sqrt{x}-\sqrt{2}}$.

Solution $\lim_{x\to 2}\frac{f(x)-f(2)}{\sqrt{x}-\sqrt{2}}\cdot\frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}}$

$$= \lim_{x\to 2}\frac{f(x)-f(2)}{x-2}\cdot\left(\sqrt{x}+\sqrt{2}\right)$$

$$= f'(2)\cdot 2\sqrt{2} = 6\sqrt{2}.$$

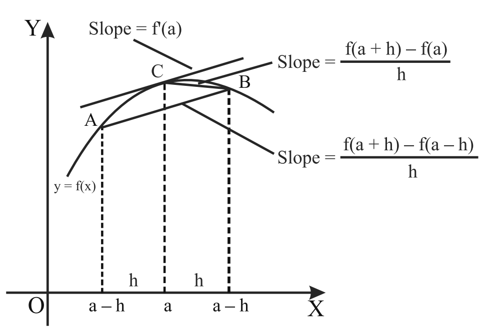
Centred difference quotient

The centred difference quotient

$$\frac{f(a+h)-f(a-h)}{2h}$$

is used to approximate $f'(a)$ in numerical work because (i) its limit as $h \to 0$ equals $f'(a)$ when $f'(a)$ exists, and (ii) it usually gives a better approximation of $f'(a)$ for a given value of h than Fermat's difference quotient

$$\frac{f(a+h)-f(a)}{h}$$



CAUTION

The quotient $\frac{f(a+h)-f(a-h)}{2h}$ may have a limit as $h \to 0$ when f has no derivative at $x = a$.

For instance, if $f(x) = |x|$ then

$$\lim_{h\to 0}\frac{|0+h|-(0-h|}{2h} = 0.$$

As we can see, the limit exists even though $f(x) = |x|$ has no derivative at $x = 0$.

In fact, $\lim_{h\to 0}\frac{f(a+h)-f(a-h)}{2h}$

$$= \frac{1}{2}\lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h}+\frac{f(a-h)-f(a)}{-h}\right]$$

$$= \frac{1}{2}(f'(a^+)+f'(a^-)).$$

We notice that it is equal to $f'(a)$, if f is differentiable at $x = a$.

Hence, if $f'(a)$ exists, then

$$\lim_{h\to 0}\frac{f(a+h)-f(a-h)}{2h} = f'(a).$$

Example 2. If $f'(5) = 7$ then find the value of

$$\lim_{t\to 0}\frac{f(5+t)-f(5-t)}{2t}.$$

Solution $\lim_{t\to 0}\frac{f(5+t)-f(5-t)}{2t}$

$$= 7.\lim_{t\to 0}\frac{f(5+t)-f(5-t)}{2t}$$

$$= \operatorname*{Limit}_{t\to 0}\left[\frac{f(5+t)-f(5)}{2t}+\frac{f(5-t)-f(5)}{-2t}\right]$$

$$= \frac{f'(5^+)}{2}+\frac{f'(5^-)}{2} = f'(5) = 7.$$

Note:

(i) If $f'(a)$ exists and $\psi(h) \to 0$ as $h \to 0$, then

$$\lim_{h\to 0}\frac{f(a+\psi(h))-f(a)}{\psi(h)} = f'(a).$$

(ii) If $f'(a)$ exists, and $\psi(h) \to 0$ and $\phi(h) \to 0$ as $h \to 0$, then

$$\lim_{h\to 0}\frac{f(a+\psi(h))-f(a+\phi(h))}{\psi(h)-\phi(h)}=f'(a).$$

Example 3. If $f'(a)=3$, evaluate the limit

$$\lim_{h\to 0}\frac{f(a+2h)-f(a-3h)}{h}.$$

Solution $\lim_{h\to 0}\frac{f(a+2h)-f(a-3h)}{h}$

$$=5.\lim_{h\to 0}\frac{f(a+2h)-f(a-3h)}{5h}$$
$$=f'(a)\times 5=3\times 5=15.$$

Example 4. If f is a differentiable function and

$$\lim_{h\to 0}\frac{f\left((\pi+h)^3\right)-f(\pi^3)}{h}=\pi,$$

then find the value of $f'(\pi^3)$.

Solution $\pi=\lim_{h\to 0}\frac{f\left((\pi+h)^3\right)-f(\pi^3)}{h}$ ...(1)

$$=\lim_{h\to 0}\frac{f\left(\pi^3+(3h^2\pi+3\pi^2h+h^3)\right)-f(\pi^3)}{h}$$

where $(3h^2\pi+3\pi^2h+h^3)\to 0$ as $h\to 0$.

$$=\lim_{h\to 0}\frac{f((\pi+h)^3)-f(\pi^3)}{(3h^2\pi+3\pi^2h+h^3)}\times\lim_{h\to 0}\frac{(3h^2\pi+3\pi^2h+h^3)}{h}$$
$$=f'(\pi^3)\cdot 3\pi^2.$$
$$\Rightarrow \pi=f'(\pi^3)\cdot 3\pi^2\Rightarrow f'(\pi^3)=\frac{1}{3\pi}.$$

Alternative :

Using L'Hospital's rule in (1), we get

$$\lim_{h\to 0}f'\left((\pi+h)^3\right)\cdot 3(\pi+h)^2=\pi$$
$$\Rightarrow f'(\pi^3)\cdot 3\pi^2=\pi$$
$$f'(\pi^3)=\frac{\pi}{3\pi^2}=\frac{1}{3\pi}.$$

Example 5. Given $f'(2)=6$ and $f'(1)=4$, evaluate

$$\lim_{h\to 0}\frac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}.$$

Solution $\lim_{h\to 0}\frac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}$

$$=\lim_{h\to 0}\frac{f(2+2h+h^2)-f(2)}{2h+h^2}\times\frac{h(2+h)}{h(1-h)}\times\frac{(1+h-h^2)-1}{f(1+h-h^2)-f(1)}$$

$$=f'(2)\times\lim_{h\to 0}\frac{2+h}{1-h}\times\frac{1}{f'(1)}=6\times 2\times\frac{1}{4}=3.$$

Example 6. If $f'(2)=4$ then find the value of

$$\lim_{h\to 0}\frac{f(2+h^2)-f(1+\cos h)}{h^2}.$$

Solution $\lim_{h\to 0}\frac{f(2+h^2)-f(1+\cos h)}{h^2}$

$$=\lim_{h\to 0}\frac{f(2+h^2)-f(2+\cos h-1)}{h^2-(\cos h-1)}\cdot\frac{h^2-(\cos h-1)}{h^2}$$

now as $h\to 0$, $h^2\to 0$ and $(\cos h-1)\to 0$

$$=f'(2)\times\lim_{h\to 0}\frac{h^2-(\cos h-1)}{h^2}$$
$$=4\times\frac{3}{2}=6.$$

Differentiability of parametric functions

Let the function $y=f(x)$ be defined by $x=x(t)$ and $y=y(t)$, where t is the parameter.

Suppose we are interested in the derivative of $y=f(x)$ and the possibility of its existence.

For parametric functions, the derivative is defined as

$$f'(x)=\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

The issue of differentiability is investigated using the existence of the limit :

$$\frac{dy}{dx}=\lim_{\delta\to 0}\frac{\frac{y(t+\delta)-y(t)}{\delta}}{\frac{x(t+\delta)-x(t)}{\delta}}=\lim_{\delta\to 0}\frac{y(t+\delta)-y(t)}{x(t+\delta)-x(t)}$$

The one-sided derivatives of f at $x=x(t)$ are given by

$$\text{L.H.D.}=f'(x^-)=\lim_{\delta\to 0}\frac{\frac{y(t-\delta)-y(t)}{-\delta}}{\frac{x(t-\delta)-x(t)}{-\delta}}$$

$$=\lim_{\delta\to 0}\frac{y(t-\delta)-y(t)}{x(t-\delta)-x(t)} \quad ...(1)$$

$$\text{R.H.D.}=f'(x^+)=\lim_{\delta\to 0}\frac{\frac{y(t+\delta)-y(t)}{\delta}}{\frac{x(t+\delta)-x(t)}{\delta}}$$

$$= \lim_{\delta \to 0} \frac{y(t+\delta) - y(t)}{x(t+\delta) - x(t)} \qquad ...(2)$$

provided x = x(t) increases with increase in t.

If x = x(t) decreases with increase in t, then (1) evaluates the R.H.D. and (2) evaluates the L.H.D.

As before, $f'(x)$ exists when both (1) and (2) exist and are equal.

Another way is to eliminate t to find the function y in terms of x and then proceed as usual.

Example 7. The function f is defined by y = f(x) where $x = 2t - |t-1|$, $y = 2t^2 - t|t|$, $t \in R$. Discuss the differentiability of y = f(x) at x = 2.

Solution We solve the equation x = 2 to get the corresponding value of t.

$$2t - |t-1| = 2 \Rightarrow t = 1.$$

We notice that at t = 1, x = x(t) increases with increase in t. Hence,

$$\text{L.H.D.} = f'(2^-) = \lim_{\delta \to 0} \frac{y(1-\delta) - y(1)}{x(1-\delta) - x(1)}$$

$$= \lim_{\delta \to 0} \frac{2(1-\delta)^2 - (1-\delta)|1-\delta| - 1}{2(1-\delta) - |1-\delta-1| - 2}$$

$$= \lim_{\delta \to 0} \frac{-2\delta + \delta^2}{-3\delta} = \frac{2}{3}.$$

$$\text{R.H.D.} = f'(2^+) = \lim_{\delta \to 0} \frac{y(1+\delta) - y(1)}{x(1+\delta) - x(1)}$$

$$= \lim_{\delta \to 0} \frac{2(1+\delta)^2 - (1+\delta)|1+\delta| - 1}{2(1+\delta) - |1+\delta-1| - 2}$$

$$= \lim_{\delta \to 0} \frac{2\delta + \delta^2}{\delta} = 2.$$

$\therefore f'(2^-) \neq f'(2^+)$ and hence $f'(2)$ does not exist.

Example 8. The function f is defined by y = f(x) where $x = 1 - t^3$, $y = [t]t^3$, $t \in R$. Discuss the differentiability of y = f(x) at x = 1.

Solution The corresponding value of t is,

$$1 - t^3 = 1 \Rightarrow t = 0$$

We notice that at t = 0, $x = 1 - t^3$ decreases with increase in t. Hence,

$$\text{R.H.D.} = f'(1^+) = \lim_{\delta \to 0} \frac{y(0-\delta) - y(0)}{x(0-\delta) - x(0)}$$

$$= \lim_{\delta \to 0} \frac{[-\delta](-\delta)^3 - 0}{1 - (-\delta)^3 - 1} = 1$$

$$\text{L.H.D.} = f'(1^-) = \lim_{\delta \to 0} \frac{y(0+\delta) - y(0)}{x(0+\delta) - x(0)}$$

$$= \lim_{\delta \to 0} \frac{[\delta]\delta^3 - 0}{1 - \delta^3 - 1} = 0.$$

Hence, f is non-differentiable at x = 1.

Example 9. The function f is defined by y = f(x) where $x = 2t - |t|$, $y = t^2 + t|t|$, $t \in R$. Discuss the continuity and differentiability at x = 0.

Solution Here, we eliminate t and get the function y = f(x) first.

If $t \geq 0$ we have $x = t, y = 2t^2$,

hence $y = 2x^2, x \geq 0$

and for $t < 0$, $x = 3t, y = 0, x < 0$

∴ The function is defined as

$$f(x) = \begin{cases} 2x^2, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{if } -1 \leq x < 0 \end{cases}$$

$$f'(0^-) = 0$$

$$f'(0^+) = \lim_{h \to 0} \frac{2h^2 - 0}{h} = 0.$$

It is clear that f(x) is differentiable and hence continuous at x = 0.

Differentiability using derivative of functions

Consider the function $f(x) = x^{4/3}e^x$. Let us find $f'(0)$ from the formula of its derivative. We differentiate f(x) using product rule (to be discussed later).

$$f'(x) = x^{4/3}e^x + \frac{4}{3}x^{1/3}e^x \qquad ...(1)$$

To find $f'(0)$, even when we donot know whether it exists, we put x = 0 into the formula (1). We get $f'(0) = 0$. This provides the result in a simple way

Now, if $f(x) = x^{1/3}\sin x$,

$$f'(x) = x^{1/3}\cos x + \frac{1}{3}\sin x \cdot x^{-2/3}$$

$$= x^{1/3}\cos x + \frac{\sin x}{3x^{2/3}} \qquad ...(2)$$

Here, $f'(x)$ is not defined at x = 0. Do we conclude that f(x) is not differentiable at x = 0 ? The answer is no. In such cases, we should try to find the derivative using its basic definition (i.e. first principles). The derivative may exist even when the formula $f'(x)$ is undefined.

$$\text{Now, } f'(0) = = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0}\frac{h^{1/3}\sinh - 0}{h} = 0.$$

$\therefore$ $f'(0) = 0$. The function f is differentiable at $x = 0$. There is another point to be noted. We find the limit of $f'(x)$ given by (2) as $x \to 0$,

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0}\left(x^{1/3}\cos x + \frac{\sin x}{3x^{2/3}}\right) = 0.$$

We are now tempted to take the limit of the derivative ($\lim_{x\to 0} f'(x)$) as the value of the derivative ($f'(0)$). Both are equal to 0 in this case.

Example 10. Suppose that the real valued function f is continuous on R and differentiable on R–{0}, and that

$$\lim_{x\to 0} f'(x) = L.$$

Prove that f is differentiable at 0 and that $f'(0) = L$.

Solution We are required to prove that

$$\lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} \text{ exists and equals L.}$$

Let $F(x) = f(x) - f(0)$ and $G(x) = x$.

Then, since f is continuous, $\lim_{x\to 0} F(x) = 0$, and of course

$\lim_{x\to 0} G(x) = 0$ too.

However, $F'(x) = f'(x)$ and $G'(x) = 1$,

so $\lim_{x\to 0}\frac{F'(x)}{G'(x)}$ exists and equals L.

By L′ Hospital′s rule, $\lim_{x\to 0}\frac{F'(x)}{G'(x)} = L$ too,

i.e. $\lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = L$, as required.

In general, the limit of the derivative may not exist, but when it exists then it is equal to the value of the derivative.

In such a case, we say that the derived function $f'(x)$ is continuous, or the function f(x) is **continuously differentiable**.

In most of the differentiable functions used in routine, the limit of the derivative exists and this gives us an opportunity to find the value of the derivative using the limit of the derivative.

But, it must be checked before differentiating the function that it is continuous.

Example 11. If $f(x)=x|x|$, then find its derivative.

Solution $f(x) = x|x|$

$$f(x) = \begin{cases} -x^2 & x < 0 \\ x^2 & x \geq 0 \end{cases} \qquad ...(1)$$

We see that f is continuous.

$$f'(x) = \begin{cases} -2x & x < 0 \\ 2x & x > 0 \end{cases} \qquad ...(2)$$

Note that the equality sign at $x = 0$ has been removed because the existence of $f'(0)$ is in doubt.

We can see that $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-}(-2x) = 0$

and $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} 2x = 0$.

Thus, at $x = 0$, L.H.D. = R.H.D. = 0,

so we have $f'(0) = 0$.

Now, we extend the definition of $f'(x)$ at $x = 0$.

Hence, $f'(x) = \begin{cases} -2x & x < 0 \\ 2x & x \geq 0 \end{cases}$

$\Rightarrow$ $f'(x) = 2|x|$.

Note:

If we consider the function

$$g(x) = \begin{cases} -x^2 & x < 0 \\ x^2 + 1 & x \geq 0 \end{cases}$$

and proceed as above, we get

$$g'(x) = \begin{cases} -2x & x < 0 \\ 2x & x > 0 \end{cases}$$

Here, also we get $\lim_{x\to 0^-} g'(x) = 0$ and $\lim_{x\to 0^+} g'(x) = 0$.

Since the function is discontinuous at $x = 0$, there is no question of differentiability. However, if the above limits are considered as the L.H.D. and R.H.D. then it would be misleading.

In fact when we calculate L.H.D. and R.H.D. using first principles, we get L.H.D. = ∞ and R.H.D. = 0.

Hence, we should check the continuity of the function in advance.

Using the following example, we now show that the derivative of a continuous function is not always a continuous function.

Example 12. If $f(x) = x^2 \sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, show that f is differentiable for every value of x but the derivative is not continuous for $x = 0$.

Solution For $x \neq 0$,

$$f'(x) = 2x \sin\frac{1}{x} + x^2 \cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$$

$$= 2x \sin\frac{1}{x} - \cos\frac{1}{x}.$$

At $x = 0$, we have

$$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{x^2 \sin\frac{1}{x}}{x}$$

$$= \lim_{x\to 0} x\sin\frac{1}{x} = 0.$$

Thus the function possesses a derivative for every value of x given by

$$f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x} \text{ when } x \neq 0,\ f'(0) = 0.$$

We now show that f' is not continuous for $x = 0$.

Here $\lim_{x\to 0} f'(x)$, i.e., $\lim_{x\to 0}\left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right)$ does not exist since $\lim_{x\to 0}\left(2x\sin\frac{1}{x}\right) = 0$ exists but $\lim_{x\to 0}\left(\cos\frac{1}{x}\right)$ does not exist.

Since, $\lim_{x\to 0} f'(x)$ does not exist, f' is discontinuous at $x = 0$. In other words, f(x) is differentiable but not continuously differentiable.

If the limit of the derivative does not exist, does it mean that the function f(x) is not differentiable ? The answer is no. This is proved by the above example.

So, the question arises that in what circumstances can we use limit of the derivative in deciding the differentiability of the function ?

Assuming that the function f(x) is continuous at $x = a$, two cases arise:

(i) If $\lim_{x\to a^-} f'(x)$ and $\lim_{x\to a^+} f'(x)$ both exist or are infinite

then $\lim_{x\to a^-} f'(x) = f'(a^-)$

i.e L.H.L. of $f'(x) =$ L.H.D. of f(x) at $x = a$

and $\lim_{x\to a^+} f'(x) = f'(a^+)$

i.e R.H.L. of $f'(x) =$ R.H.D. of f(x) at $x = a$.

(a) If $\lim_{x\to a^-} f'(x) = \lim_{x\to a^+} f'(x)$, then f(x) is differentiable and its derivative $f'(x)$ is continuous at $x = a$.

Example 13.

If $f(x) = \begin{cases} x\tan^{-1}x + \sec^{-1}(1/x), & x \in (-1,1)-\{0\} \\ \pi/2 & \text{if } x = 0 \end{cases}$,

then find $f'(0)$, if it exists.

Solution

$$f'(x) = \frac{x}{1+x^2} + \tan^{-1}x + \frac{1}{\left|\frac{1}{x}\right|\sqrt{\frac{1}{x^2}-1}}\left(-\frac{1}{x^2}\right)$$

$$= \frac{x}{1+x^2} + \tan^{-1}x + \frac{|x|^2}{\sqrt{1-x^2}}\left(-\frac{1}{x^2}\right)$$

$$\lim_{x\to 0^-} f'(x) = -1 \text{ and } \lim_{x\to 0^+} f'(x) = -1$$

Hence $f'(0) = -1$

Alternative :

Using first principles :

$$f'(0^+) = \lim_{h\to 0}\frac{h\tan^{-1}(h) + \sec^{-1}(1/h) - \pi/2}{h}$$

$$= \lim_{h\to 0}\frac{\cos^{-1}(h) - \pi/2}{h} = \lim_{h\to 0}\frac{-\sin^{-1}(h)}{h} = -1$$

Similarly, $f'(0^+) = -1$

Hence, $f'(0) = -1$.

(b) If $\lim_{x\to a^-} f'(x) \neq \lim_{x\to a^+} f'(x)$, then f(x) is non-differentiable and its derivative $f'(x)$ is discontinuous at $x = a$.

Example 14. Let $f(x) = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$. Define $f'(x)$ stating clearly the points where f(x) is not differentiable.

Solution $f(x) = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$, $x \in R$

$$f'(x) = -\frac{1}{\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}}\cdot\frac{2(1+x^2)-4x^2}{(1+x^2)^2}$$

$$f'(x) = -\frac{2(1-x^2)}{|1-x^2|(1+x^2)}$$

$$\lim_{x\to 1^-} f'(x) = -1 \text{ and } \lim_{x\to 1^+} f'(x) = 1$$

Since $\lim_{x\to 1^-} f'(x)$ and $\lim_{x\to 1^+} f'(x)$ are unequal, $f'(1)$ does not exist.

Similarly $f'(-1)$ does not exist.

Hence, $f'(x) = \begin{cases} -\dfrac{2}{1+x^2} & \text{if } -1 < x < 1 \\ \text{non existent} & \text{if } x = \pm 1 \\ \dfrac{2}{1+x^2} & \text{if } x > 1 \text{ or } x < -1 \end{cases}$

(c) If $\lim_{x \to a^-} f'(x)$ and $\lim_{x \to a^+} f'(x)$ are infinite, then f(x) has an infinite derivative at x = a and hence it is non-differentiable there and f ′(x) is discontinuous at x = a.

Example 15. Suppose $f(x) = \begin{bmatrix} |x|^x, & x \neq 0 \\ 1, & x = 0 \end{bmatrix}$, then find whether f(x) is differentiable at x = 0.

Solution

For $x > 0, f(x) = x^x$

and $f'(x) = x^x(1 + \ln x)$ ...(1)

For $x < 0$, $f(x) = (-x)^x = e^{x \ln(-x)}$

and $f'(x) = (-x)^x[1 + \ln(-x)]$ ...(2)

It can be shown using L'Hospital's rule that

$$\lim_{x \to 0} x^x = 1 \quad \text{and} \quad \lim_{x \to 0} (-x)^x = 1$$

Now, $\lim_{x \to 0^-} f'(x) = -\infty$ and $\lim_{x \to 0^+} f'(x) = -\infty$

$$\lim_{x \to 0} f'(x) = -\infty$$

Hence f is not differentiable at x = 0.

(ii) If any of the limits $\lim_{x \to a^-} f'(x)$ or $\lim_{x \to a^+} f'(x)$ does not exist, then we cannot conclude anything about the differentiability of the function. In such a case, we should try to find the derivative using its basic definition (i.e. first principles).

In the previous example, we have seen this situation :

$f(x) = x^2 \sin(1/x)$ when $x \neq 0$ and $f(0) = 0$,

For $x \neq 0$, $f'(x) = 2x \sin \dfrac{1}{x} - \cos \dfrac{1}{x}$.

Here $\lim_{x \to 0} f'(x) = \lim_{x \to 0} \left(2x \sin \dfrac{1}{x} - \cos \dfrac{1}{x}\right)$ does not exist.

However, using first principles, we have

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h}}{h}$$

$$= \lim_{h \to 0} h \sin \frac{1}{h} = 0.$$

Hence, the function f is differentiable at x = 0, even if the limit of its derivative does not exist.

Example 16. Calculate the derivative of the function

$$f(x) = \sqrt{x^2 - 2x + 1} \text{ on the interval } [0, 2].$$

Solution Since the expression under the radical sign is a perfect square, we can, according to the definition of the modulus, represent the given function in the following form :

$$f(x) = |x - 1| = \begin{cases} 1 - x, & 0 \le x < 1, \\ x - 1, & 1 \le x \le 2 \end{cases}$$

Differentiating f(x) separately on the intervals [0, 1) and (1, 2], we obtain

$$f'(x) = \begin{cases} -1, & 0 \le x < 1, \\ 1, & 1 < x \le 2 \end{cases}$$

$$\lim_{x \to 1^-} f'(x) = -1 \text{ and } \lim_{x \to 1^+} f'(x) = 1$$

Since the left and right derivatives of f (x) at the point x = 1 do not coincide, the derivative does not exist at x = 1. We take the values of the left hand derivative of function at the point 2 and of its right hand derivative at the point 0 as the values of f′(x) at the endpoints of the interval [0, 2].

Thus $f'(x) = \begin{cases} -1, & x \in [0, 1), \\ 1, & x \in (1, 2] \end{cases}$

Example 17. Check the differentiability of the function $f(x) = \sqrt{1 - \sqrt{1 - x^2}}$.

Solution The function $f(x) = \sqrt{1 - \sqrt{1 - x^2}}$ is defined for $x \in [-1, 1]$. It is continuous in its domain. Differentiating w.r.t. x, we have

$$f'(x) = \frac{1}{2\sqrt{1 - \sqrt{1 - x^2}}} \cdot \frac{-1}{2\sqrt{1 - x^2}} \cdot -2x$$

$$= \frac{x}{2\sqrt{1 - x^2}\sqrt{1 - \sqrt{1 - x^2}}}$$

The formula of f′(x) is undefined at x = – 1, 0, 1.

The differentiability of the function should be checked at x = – 1, 0, 1.

We can see that

$$\lim_{x \to -1^+} f'(x) = -\infty \text{ and } \lim_{x \to -1^-} f'(x) = \infty.$$

Hence, f has an infinite derivative at x = – 1 and 1.

To check the existence of f′(x) at x = 0, we find whether $\lim_{x \to 0} f'(x)$ exists or not.

We have $\lim_{x\to 0}\frac{x}{\sqrt{1-x^2}\sqrt{1-\sqrt{1-x^2}}}=\lim_{x\to 0}\frac{x}{\sqrt{1-\sqrt{1-x^2}}}$

$=\lim_{x\to 0} x\sqrt{\frac{1+\sqrt{1-x^2}}{x^2}}$

$=\lim_{x\to 0}\frac{\sqrt{2}x}{|x|}=\begin{cases}-\sqrt{2} & \text{as } x\to 0^- \\ \sqrt{2} & \text{as } x\to 0^+\end{cases}$

= does not exist.

Hence, f is non-differentiable at $x = 0$.

Thus, the function f is differentiable in its domain except at $x = -1, 0, 1$.

Alternative approach at $x = 0$ using first principles.

$$f'(0^-)=\lim_{h\to 0}\left\{\frac{f(0-h)-f(0)}{-h}\right\}$$

$$=\lim_{h\to 0}\left\{\frac{\sqrt{1-\sqrt{(1-h^2)}}-0}{-h}\right\}$$

$$=\lim_{h\to 0}\frac{\sqrt{1-\sqrt{(1-h^2)}}}{-h}\times\frac{\sqrt{1+(1-h^2)}}{1+\sqrt{1-h^2}}$$

$$=\lim_{h\to 0}\frac{h}{(-h)\sqrt{1+\sqrt{(1-h^2)}}}=-\frac{1}{\sqrt{2}}$$

and $f'(0^+)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$

$$=\lim_{h\to 0}\frac{\sqrt{1-\sqrt{(1-h^2)}}-0}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{1-\sqrt{(1-h^2)}}}{h}\times\frac{\sqrt{1+(1-h^2)}}{\sqrt{1+\sqrt{1-h^2}}}$$

$$=\lim_{h\to 0}\frac{h}{h\sqrt{1+\sqrt{(1-h^2)}}}=\frac{1}{\sqrt{2}}$$

$\because$ $f'(0^-)\neq f'(0^+)$, f(x) is not differentiable at $x = 0$.

Example 18. If $f(x)=\begin{cases}A+Bx^2 & , x<1 \\ 3Ax-B+2, & x\geq 1\end{cases}$ find the value of A and B so that f(x) is differentiable at $x = 1$.

Solution Since f(x) will be differentiable at $x = 1$, it must be continuous,

$\therefore$ $3A-B+2=A+B$

$\therefore$ $A-B+1=0$ ...(1)

Now, $f'(x)=\begin{cases}2Bx, & x<1 \\ 3A, & x>1\end{cases}$

$\lim_{x\to 1^-} f'(x)=2B$ and $\lim_{x\to 1^+} f'(x)=3A$

$\therefore$ $3A=2B$ ...(2)

From (1) and (2), $\frac{2B}{3}-B+1=0$

$\Rightarrow$ $-\frac{B}{3}=-1$ $\Rightarrow$ $B = 3$

$\Rightarrow$ $A=2$.

Example 19. If $f(x)=\begin{cases}|1-4x^2| & , 0\leq x<1 \\ [x^2-2x] & , 1\leq x<2\end{cases}$

where [.] denotes the greatest integer function.

Discuss the continuity and differentiability of f(x) in [0, 2).

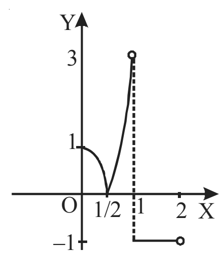
Solution Since $1\leq x<2$ we have $0\leq x-1<1$,

and $[x^2-2x]=[(x-1)^2-1]=[(x-1)^2]-1$

$=0-1=-1$

$\therefore$ $f(x)=\begin{cases}1-4x^2 & , 0\leq x<\frac{1}{2} \\ 4x^2-1 & , \frac{1}{2}\leq x<1 \\ -1 & , 1\leq x<2\end{cases}$

Hence, the graph of f(x) is :



It is clear from the graph that f(x) is discontinuous at $x = 1$ and not differentiable at $x=\frac{1}{2}$ and $x = 1$. To verify these details we calculate the derivative of f :

$$f'(x)=\begin{cases}-8x, & 0\leq x<1/2 \\ 8x, & 1/2<x<1 \\ 0, & 1<x<2\end{cases}$$

This shows that f(x) is not differentiable

at $x = 1/2$ (as RHD = 4 and LHD = –4)

and $x = 1$ (as RHD = 0 and LHD = 8)

Therefore, f(x) is differentiable $\forall$ $x\in[0,2)-\{1/2,1\}$.

Concept Problems

D

1. Evaluate $\lim\limits_{x \to 0} \dfrac{\sin(3+x)^2 - \sin 9}{x}$.
2. Find a function f and a number a such that $\lim\limits_{h \to 0} \dfrac{(2+h)^6 - 64}{h} = f'(a)$.
3. Each of the following limit represents the derivative of some function f at some number a. State such an f and a in each case.

   (i) $\lim\limits_{h \to 0} \dfrac{2^x - 32}{x-5}$ (ii) $\lim\limits_{h \to 0} \dfrac{\cos(\pi + h) + 1}{h}$
4. Let $f(x) = \begin{cases} x^2 & x \le 0 \\ -x^2 & x > 0 \end{cases}$.

   Prove that f is differentiable. Is f' is continuous?
5. If $f'(a) = \dfrac{1}{4}$, find $\lim\limits_{h \to 0} \dfrac{f(a+2h^2) - f(a-2h^2)}{h^2}$.
6. If $f'(2) = 4$ then find the value of $\lim\limits_{h \to 0} \dfrac{f(2+h) - f(2+\sin h)}{h.\sin h.\tan h}$.
7. Discuss the differentiability of the function y=f(x) defined as $x = 2t + |t|$ and $y = [t]t^2$ at $x = 0$.
8. Let $f(x) = \begin{cases} 3x^2, & x \le 1 \\ ax + b, & x > 1 \end{cases}$. Find the values of a and b so that f will be differentiable at x = 1.
9. Let $f(x) = \begin{cases} x^2, & x \le 0 \\ x^2 + 1, & x > 0 \end{cases}$. Show that $\lim\limits_{x \to 0^-} f'(x) = \lim\limits_{x \to 0^+} f'(x)$ but that $f'(0)$ does not exist.

3.8 DERIVATIVES OF HIGHER ORDER

Because the derivative of a function is a function, differentiation can be applied over and over, as long as the derivative itself is a differentiable function.

Let a function y = f(x) be defined on an open interval (a, b). Its derivative, if it exists on (a, b) is a certain function $f'(x)$ [or (dy/dx) or y] and is called the first derivative of y w. r. t. x.

If it happens that the first derivative has a derivative on (a, b) then this derivative is called the second derivative of y w. r. t. x and is denoted by $f''(x)$ or (d^2y/dx^2) or y''.

Once we have found the derivative f' of any function f, we can go on and find the derivative of f'.

$$f''(a) = \lim_{h \to 0} \frac{f'(a+h) - f'(a)}{h}.$$

Similarly, the third order derivative of y w. r. t. x, if it exists, is defined by $\dfrac{d^3y}{dx^3} = \dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right)$. It is also denoted by $f'''(x)$ or y'''.

The third derivative, written f''', is the derivative of the second derivative, and, in principle, we can go on forever and form derivatives of higher order. We adopt the alternative notation $f^{(n)}$ for the n^{th} derivative of f. Notice also that for derivatives higher than the third, the parentheses distinguish a derivative from a power. For example, $f^4 \ne f^{(4)}$.

Twice differentiability

Suppose that a function y = f(x) defined in an interval has the derivative $f'(x)$ with respect to the independent variable x. If the function $f'(x)$ is further differentiable, its derivative is called the second derivative (or the derivative of the second order) of the original function f(x) and is denoted $f''(x)$.

A function f(x) is twice differentiable at x = a if its derivative $f'(x)$ is differentiable at x = a i.e.

the limit $f''(a) = \lim\limits_{h \to 0} \dfrac{f'(a+h) - f'(a)}{h}$ exists.

Alternatively, $f''(a) = \lim\limits_{x \to a} \dfrac{f'(x) - f'(a)}{x-a}$.

The value $f''(x)$ of the second derivative at a point x characterizes the rate of change of $f'(x)$ at that point, that is the rate of change of the rate of change of f(x). By analogy with mechanics, we can say that f"(x) is the acceleration of the change of the function f(x) at the given point x.

The geometrical meaning of the second derivative will be discussed later.

Further, the n^{th} derivative of f(x) at x = a exists if

$$f^{(n)}(a) = [f^{(n-1)}]'(a) = \lim_{x \to a} \frac{f^{(n-1)}(x) - f^{(n-1)}(a)}{x-a} \text{ exists.}$$

Example 1. Find whether the function $f(x) = x^{4/3} \sin x$ is twice differentiable at $x = 0$.

Solution Let us find $f'(x)$ first. We differentiate f(x) using product rule.

$$f'(x) = x^{4/3}\cos x + \frac{4}{3}x^{1/3}\sin x \quad ...(1)$$

Now, $f''(0) = \lim_{h\to 0}\frac{f'(0+h) - f'(0)}{h}$

$$= \lim_{h\to 0}\frac{h^{4/3}\cos h + \frac{4}{3}h^{1/3}\sin h - 0}{h}$$

$$= \lim_{h\to 0}\left(h^{1/3}\cos h + \frac{4h^{1/3}\sin h}{3h}\right) = 0.$$

$\therefore \quad f''(0) = 0.$

Hence, the function f is twice differentiable at $x = 0$.

Example 2. Find whether

$$f(x) = \begin{cases} -x^2 + x & x < 0 \\ \sin x + x^2 & x \ge 0 \end{cases}$$

is twice differentiable and find its second derivative.

Solution $f(x) = \begin{cases} x^2 + x & x < 0 \\ \sin x + x^2 & x \ge 0 \end{cases}$ ...(1)

We see that f is continuous.

$$f'(x) = \begin{cases} 2x + 1 & x < 0 \\ \cos x + 2x & x > 0 \end{cases} \quad ...(2)$$

Note that the equality sign at $x = 0$ has been removed because the existence of $f'(0)$ is in doubt.

We can see that $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x) = 1$.

So we have $f'(0) = 1$.

Now, we extend the definition of $f'(x)$ at $x = 0$.

Hence, $f'(x) = \begin{cases} 2x + 1 & x < 0 \\ \cos x + 2x & x \ge 0 \end{cases}$

Now $f''(x) = \begin{cases} 2 & x < 0 \\ -\sin x + 2 & x > 0 \end{cases}$

We find that $\lim_{x\to 0^-} f''(x) = \lim_{x\to 0^+} f''(x) = 2$.

$\therefore$ $f''(0) = 2$ and the function f is twice differentiable at $x = 0$. We can easily see that f is twice differentiable at other values of x.

Example 3. Let f be defined in a neighbourhood of x and $f''(x)$ exists.

Show that $\lim_{h\to 0}\frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x)$

Show by an example that the limit may exist ever if $f''(x)$ does not exist.

Solution $f''(x) = \lim_{h\to 0}\frac{f'(x+h) - f'(x)}{h}$

$$= \lim_{h\to 0}\frac{\lim_{k\to 0}\left[\frac{f(x+h+k) - f(x+h)}{k} - \frac{f(x+k) - f(x)}{k}\right]}{h}$$

Let $k = -h$

$$\Rightarrow \quad f''(x) = -\lim_{h\to 0}\frac{f(x) - f(x+h) - 2(x-h) + f(x)}{h^2}$$

$$= \lim_{h\to 0}\frac{f(x+h) + f(x-h) - 2f(x)}{h^2}$$

Note: The above limit cannot be used as the test for twice differentiability of f. One can verify this by using $f(x) = x|x|$.

Let us calculate the limit first :

$$\lim_{h\to 0}\frac{f(0+h) + f(0-h) - 2f(0)}{h^2}$$

$$= \lim_{h\to 0}\frac{h|h| + (-h)|-h| - 0}{h^2} = \lim_{h\to 0}\frac{0}{h^2} = 0.$$

In the previous section we had calculated

$$f'(x) = \begin{cases} -2x & x < 0 \\ 2x & x \ge 0 \end{cases}$$

from where $f''(0^-) = -2$ and $f''(0^+) = 2$.

Hence $f''(0)$ does not exist. However, the given limit exists.

Example 4. If $f(x) = \begin{cases} xe^x & x \le 0 \\ x + x^2 - x^3 & x > 0 \end{cases}$ then examine whether $f'(x)$ is continuous and differentiable at $x = 0$.

Solution $f(x) = \begin{cases} xe^x & x \le 0 \\ x + x^2 - x^3 & x > 0 \end{cases}$

f(x) is continuous at $x = 0$.

$$f'(x) = \begin{cases} xe^x + e^x, & x < 0 \\ 1 + 2x - 3x^2, & x > 0 \end{cases}$$

$$\lim_{x\to 0^-} f'(x) = 1 = \lim_{x\to 0^+} f'(x)$$

Hence f(x) is differentiable at x = 0 and thus

$$f'(x)=\begin{cases} xe^x+e^x, & x\le 0 \\ 1+2x-3x^2, & x>0 \end{cases} \text{ is continuous.}$$

Now, $f''(x)=\begin{cases} e^x+(x+1)e^x, & x<0 \\ 2-6x, & x>0 \end{cases}$

$$\lim_{x\to 0^+} f''(x)=\lim_{x\to 0^-} f''(x)=2$$

$\Rightarrow$ f′(x) is also differentiable at x = 0.

Example 5. Let $f(x)=\begin{cases} x^3 & \text{if } x<1 \\ ax^2+bx+c & \text{if } x\ge 1 \end{cases}$.

If f″ (1) exist then find the value of $a^2+b^2+c^2$.

Solution For continuity at x = 1 we must have

$f(1^+)=f(1^-)$

$\Rightarrow$ $a+b+c=1$ ...(1)

$$f'(x)=\begin{cases} 3x^2 & \text{if } x<1 \\ 2ax+b & \text{if } x>1 \end{cases}$$

For continuity of f′(x) at x = 1

$f'(1^+)=f'(1^-)$ 2a + b

$\Rightarrow$ $2a+b=3$ ...(2)

$$f''(x)=\begin{cases} 6x & \text{if } x<1 \\ 2a & \text{if } x>1 \end{cases}$$

$f''(1^-)=6,\quad f''(1^+)=2a$

Hence, $2a=6 \quad\Rightarrow a=3$

From (1) and (2), $b=-3,\quad c=1$

Hence, the value of $a^2+b^2+c^2=19$.

Example 6. Let $f(x)=\begin{cases} 2x^2\sin\pi x & x\le 1 \\ x^3+ax^2+b & x>1 \end{cases}$

be a differentiable function . Examine whether it is twice differentiable in R .

Solution For continuity at x = 1 we must have

$a+b+1=0$

$$f'(x)=\begin{cases} 4x\sin\pi x+2\pi x^2\cos\pi x & x<1 \\ 3x^2+2ax & x>1 \end{cases}$$

For differentiability of f :

$f'(1^+)=3+2a,\ f'(1^-)=-2\pi$ should be equal.

$$\Rightarrow\quad a=-\frac{2\pi+3}{2} \text{ and } b=\frac{2\pi+3}{2}$$

Thus $f'(x)=\begin{cases} 4x\sin\pi x+2\pi x^2\cos\pi x & ,x\le 1 \\ 3x^2-(3+2\pi)x & ,x>1 \end{cases}$ and

$$f''(x)=\begin{cases} (4-2\pi^2x^2)\sin\pi x+8\pi x\cos\pi x & ,\ x<1 \\ 6x-3-2\pi & ,\ x>1 \end{cases}$$

$f''(1^-)=-8\pi$ and $f''(1^+)=3-2\pi$

$\Rightarrow$ f″(x) does not exist at x = 1.

However, f(x) is twice differentiable for x other than x = 1

Example 7. Find

$$f''(x) \text{ if } f(x)=\begin{cases} x^3\sin(1/x) & \text{for } x\ne 0, \\ 0 & \text{for } x=0 \end{cases}$$

and find whether f″(x) is continuous at the point x = 0.

Solution For x ≠ 0 the derivative f ′(x) can be found by differentiating the function $x^3\sin(1/x)$ according to the rule of differentiation of a product. This yields

$f'(x)=3x^2\sin(1/x)-x\cos(1/x)\ (x\ne 0)$.

$$\lim_{x\to 0^-} f'(x)=0=\lim_{x\to 0^+} f'(x)$$

This means that f′(0) exists

Thus f′(x) exists at all points :

$$f'(x)=\begin{cases} 3x^2\sin(1/x)-x\cos(1/x) & \text{for } x\ne 0, \\ 0 & \text{for } x=0. \end{cases}$$

Now, $f''(x)=6x\sin(1/x)-4\cos(1/x)$
$-(1/x)\sin(1/x)\qquad (x\ne 0)$.

$\lim_{x\to 0} f''(x)$ does not exist.

To test whether f″(x) exists at x = 0, we apply first principles on f ′(x)

$$f''(x)=\lim_{h\to 0}\frac{f'(h)-f'(0)}{h}$$

$$=\lim_{h\to 0}\frac{3h^2\sin\frac{1}{h}-h\cos\frac{1}{h}-0}{h}$$

$$=\lim_{h\to 0}\left(3h\sin\frac{1}{h}-\cos\frac{1}{h}\right)$$

which does not exist.

Thus f is twice differentiable for all x except x = 0. Further, f″(x) is discontinuous at the point x = 0.

Example 8. Let f, g be twice differentiable functions from R to R, and suppose that f(x) = g(x)/x for all x ≠ 0. Given that g(0) = g ′(0) = 0 and that g"(0) = 6, determine f(0) and f′(0).

Solution By the definition of g ′(0), we have

$$0=g'(0)=\lim_{x\to 0}\frac{g(x)-g(0)}{x-0}$$

$= \lim_{x\to 0} \frac{g(x)}{x}$ (since $g(0) = g'(0) = 0$)

$= \lim_{x\to 0} f(x)$ (since $f(x) = \frac{g(x)}{x}$ for $x \neq 0$),

$= f(0)$ (since f is continuous)

So $f(0) = 0$. To find $f'(0)$, note that

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{g(x)}{x^2}$$

(since $f(0) = 0$ and $f(x) = \frac{g(x)}{x}$ for $x \neq 0$).

To determine this latter limit, we can use L′Hospital′s rule. Certainly $\lim_{x\to 0} g(x) = 0$ and $\lim_{x\to 0} x^2 = 0$.

So we look at $\frac{g'(x)}{2x}$ (differentiating top and bottom).

But $g'(0) = 0$, so $\lim_{x\to 0} \frac{g'(x)}{x} = g''(0) = 6$, and so

$\lim_{x\to 0} \frac{g'(x)}{2x} = 3$. Thus by L′Hospital′s rule,

$\lim_{x\to 0} \frac{g(x)}{x^2} = 3$ too, and this gives $f'(0) = 3$.

Concept Problems E

1. Let $f(x) = \begin{cases} x^2, & x \le 0 \\ x^3, & x > 0 \end{cases}$ Show that $f'(0)$ exists but $f''(0)$ does not.

2. Let f be the function defined by

$$f(x) = \begin{cases} \frac{x^2}{2} & \text{if } x \ge 0, \\ -\frac{x^2}{2} & \text{if } x \le 0. \end{cases}$$

(a) Compute f'.
(b) Is f a differentiable function?
(c) Show that $f''(0)$ does not exist, and compute $f''(x)$ for $x \neq 0$.

3. Let $f(x) = x^{4/3}$
(a) Compute f'.
(b) Is f a differentiable function?
(c) Show that $f''(0)$ does not exist, and compute $f''(x)$ for $x \neq 0$.

4. Let the function g be defined by

$$g(x) = \begin{cases} x^2 & x \le 0, \\ 2x - 1 & x > 1. \end{cases}$$

(a) Compute g' and g''.
(b) Are g and g' differentiable functions ?

5. Evaluate $\lim_{x\to 0} \frac{\sin(a + 2x) - 2\sin(a + x) + \sin a}{x^2}$.

Practice Problems D

6. Find y" for the following functions :
(a) $y = |x^3|$
(b) $y = \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0 \end{cases}$

Also find y" (0) if it exists.

7. A function f is defined as

$$f(x) = \begin{cases} \frac{1}{2}(b^2 - a^2) & \text{for } 0 \le x \le a \\ \frac{1}{2}b^2 - \frac{x^2}{6} - \frac{a^2}{3} & \text{for } a < x \le b \\ \frac{1}{2}\left(\frac{b^3 - a^3}{x}\right) & \text{for } x > b \end{cases}$$

Prove that f and f' are continuous but f'' is discontinuous.

8. Evaluate $\lim_{x\to 0} \frac{\tan(a + 2x) - 2\tan(a + x) + \tan a}{x^2}$.

9. (a) Show that $f(x) = x^{7/3}$ is twice differentiable at 0, but not three times differentiable at 0.
(b) Find an exponent k such that $f(x) = x^k$ is $(n - 1)$ times differentiable at 0, but not n times differentiable at 0.

3.9 ALGEBRA OF DIFFERENTIABLE FUNCTIONS

We can deduce from the theorems on limits that the sum, product, difference or quotient of two functions which are differentiable at a certain point are themselves differentiable at that point (except that, in the case of the quotient, the denominator must not vanish at the point in question). Further it is true that composition of a differentiable function with a differentiable function is a differentiable function.

1. If f(x) and g(x) are differentiable at x = a, then the following functions are also differentiable at x = a.
 (i) cf(x) is differentiable at x = a, where c is any constant.
 (ii) f(x) ± g(x) is differentiable at x = a.
 (iii) f(x). g(x) is differentiable at x = a.
 (iv) f(x)/g(x) is differentiable at x = a, provided $g(a) \neq 0$.

Here, we prove some of these results :

Theorem If f and g are differentiable at x = a, then so is their product f + g.

Proof We apply the definition of derivative to F(x) = f(x) + g(x) as follows :

$$F'(a) = \lim_{h\to 0}\frac{(f(a+h)+g(a+h))-(f(a)+g(a))}{h}$$
$$= \lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h}+\frac{g(a+h)-g(a)}{h}\right]$$
$$= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}+\lim_{h\to 0}\frac{g(a+h)-g(a)}{h}$$
$$= f'(a) + g'(a).$$

Therefore, f(x) + g(x) is differentiable at x = a.

Note: The sum of a finite number of functions differentiable at a point is a differentiable function at the point.

Theorem If f and g are differentiable at x = a, then so is their product f. g.

Proof Let F(x) = f(x). g(x)

$$F'(a) = \lim_{h\to 0}\frac{f(a+h)g(a+h)-f(a)g(a)}{h}$$
$$= \lim_{h\to 0}\frac{f(a+h)g(a+h)-f(a+h)g(a)+f(a+h)g(a)-f(a)g(a)}{h}$$
$$= \lim_{h\to 0}\left[f(a+h)\frac{g(a+h)-g(a)}{h}+g(a)\frac{f(a+h)-f(a)}{h}\right]$$
$$= \lim_{h\to 0}f(a+h)\,.\,\lim_{h\to 0}\frac{g(a+h)-g(a)}{h} + g(a)\,.\,\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

As h approaches zero, f(a+h) approaches f(a) because f, being differentiable at a, is continuous at a.

$$= f(a)\,g'(a) + f'(a)\,g(a).$$

Therefore, f(x). g(x) is differentiable at x = a.

Note: The product of a finite number of functions differentiable at a point is a differentiable function at that point.

By the above theorems we recognize that the functions

$y = x^2 + 3x + 2$
$y = \sin^3 x - x\sin x - (x^4 - 1)\cos x$, and
$y = (\sin x - 2x)/(x^2 + 1)$

are differentiable at every point in a domain common to all functions involved.

2. If f(x) is differentiable at x = a and g(x) is non-differentiable at x = a, then we have the following results:
 (i) Both the functions f(x) + g(x) and f(x) – g(x) are non-differentiable at x = a.
 For example, consider, f(x) = x and g(x) = |x|. Here f(x) is differentiable at x = 0 and g(x) is non-differentiable at x = 0. Both the sum function x + |x| and the difference function x – |x| are non-differentiable at x = 0.
 (ii) f(x). g(x) is not necessarily non-differentiable at x = a. We need to find the result by first principles.
 For example, consider, $f(x) = x^3$ and g(x) = sgn(x).
 Here f(x) is differentiable at x = 0 and g(x) is non-differentiable at x = 0. But the product function

$$f(x)g(x) = \begin{cases} x^3, & x > 0 \\ 0, & x = 0 \\ -x^3, & x < 0 \end{cases}$$

 is differentiable at x = 0.
 As another example, the product of the functions

$$f(x) = x \quad\text{and}\quad g(x) = \begin{cases} x\sin\dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

 is differentiable at x = 0 even when f(x) is differentiable at x = 0 and g(x) is non-differentiable at x = 0.
 However, the product of the functions f(x) = x and g(x) = [x] is non-differentiable at x = 1.
 (iii) f(x)/g(x) is not necessarily non-differentiable at x = a. Here also we need to work on the function f(x)/g(x) to get the result.

Let $f(x) = x^2(x^2 - 1)$ and $g(x) = \begin{cases} x+1, & x \ge 0 \\ x-1, & x < 0 \end{cases}$.

Here f(x) is differentiable at x = 0 and g(x) is non-differentiable at x = 0.

$$f(x)/g(x) = \begin{cases} x^2(x-1), & x \ge 0 \\ x^2(x+1), & x < 0 \end{cases}$$

We can check that f(x)/g(x) is differentiable at x = 0.

Example 1. Discuss the differentiability of $f(x) = [x] + |2x - 1|$.

Solution Let us study the functions y = [x] and $y = |2x - 1|$. It is a clear that f(x) = [x] is non-differentiable at all integral points and $g(x) = |2x - 1|$ is differentiable for all $x \in R - \{1/2\}$. The sum of a non-differentiable and a differentiable function is non-differentiable. Hence f(x) + g(x) is non-differentiable at all integral points and x = 1/2.

3. If f(x) and g(x) both are non-differentiable at x = a, then we have the following results.

(i) The functions f(x) + g(x) and f(x) – g(x) are not necessarily non-differentiable at x = a. However, atmost one of f(x) + g(x) or f(x) – g(x) can be differentiable at x = a. That is, both of them cannot be differentiable simultaneously at x = a.

Let us assume that both f(x) + g(x) and f(x) – g(x) are differentiable. Then the sum of functions (f(x) + g(x)) + (f(x) – g(x)) = 2f(x) must be differentiable at x = a, which is wrong as it is given that f(x) is non-differentiable at x = a. Hence our assumption is wrong. So, both the functions cannot be differentiable simultaneously at x = a.

For example, consider, f(x) = [x] and g(x) = {x}. Here both f(x) and g(x) are non-differentiable at x = 0.

The sum function [x] + {x} being equal to x is differentiable at x = 0. The difference function [x] – {x} however is non-differentiable at x = 0. But this does not mean that one of the functions f(x) + g(x) or f(x) – g(x) must be differentiable. We can have both the functions non-differentiable. For example, if f(x) = 2[x] and g(x) = {x}, then both the functions f(x) + g(x) or f(x) – g(x) are non-differentiable simultaneously at x = 0.

(ii) f(x). g(x) is not necessarily non-differentiable at x = a. We need to find the result by applying first principles on the product f(x). g(x).

For example, consider, f(x) = [x] and g(x) = [–x]. Here both f(x) and g(x) are non-differentiable at x = 0 but, the product function [x].[–x] is differentiable at x = 0.

Further, f(x) = [x] and g(x) = {x} are both non-differentiable at x = 0 and, the product function [x].{x} is non-differentiable at x = 0. Hence, we cannot comment in advance when both the functions are non-differentiable at x = a.

(iii) f(x)/g(x) is not necessarily non-differentiable at x = a. Here also we need to work on the function f(x)/g(x) to get the result.

Let $f(x) = \begin{cases} x^2 - 1, & x \ge 0 \\ x+1, & x < 0 \end{cases}$ and

$g(x) = \begin{cases} x+1, & x \ge 0 \\ x-1, & x < 0 \end{cases}$.

Here both f(x) and g(x) are non-differentiable at x = 0. But we find that f(x)/g(x) is differentiable at x = 0.

Differentiability of Composite Functions

Theorem If f(x) is differentiable at x = a and g(x) is differentiable at x = f(a) then the composite function (gof)(x) is differentiable at x = a.

Note: A function of a function composed of a finite number of differentiable functions is a differentiable function.

For example, f(x) = sin x is differentiable at $x = \frac{\pi}{2}$ and

$g(x) = \begin{cases} x^2 - 1, & x \le 1 \\ 2x - 2, & x > 1 \end{cases}$ is differentiable at $x = f(\frac{\pi}{2}) = 1$.

Hence the composite function (gof)(x) is differentiable at $x = \frac{\pi}{2}$.

Note:

1. Let a function f(x) be differentiable at all points in the interval [a, b], and let its range be the interval [A, B] and further a function g(x) is differentiable in the interval [A, B], then the composite function (gof)(x) is differentiable in the interval [a, b].

2. If the function f is differentiable everywhere and the function g is differentiable everywhere, then the composition gof is differentiable everywhere.
3. If f(x) is differentiable, then $|(f(x))^p|$, $p > 1$ is also differentiable.

For example, $f(x) = \dfrac{\sin x}{x^2+1}$ and $g(x) = x|x|$ are differentiable for all x. Hence, the composite function $(gof)(x) = \dfrac{\sin x}{x^2+1}\left|\dfrac{\sin x}{x^2+1}\right|$ is also differentiable for all x.

Example 2. Check the differentiability of

$$f(x) = \frac{x^3}{\sin(\cos x)}$$

Solution The numerator is differentiable for all x. As far as the denominator is concerned, according to the theorem on differentiability of a composite function, it is differentiable at points where the function u = cos x is differentiable, since the function sin u is differentiable everywhere.

We must exclude the points at which sin(cos x) = 0. i.e. the points at which $\cos x = k\pi$ ($k \in I$), or cos x = 0. Thus, the function f(x) is differentiable everywhere except at the points $x = (2n + 1)\pi/2$ ($n \in I$).

Example 3. What can you say about the differentiability of the function $f(x) = \sqrt{9-x^2}$?

Solution Because the natural domain of this function is the closed interval [–3, 3], we will need to investigate the differentiability of f on the open interval (–3, 3) and at the two endpoints. If c is any number in the interval (–3, 3), then f is differentiable at c. The function f is non-differentiable at the endpoints since

$$f'(3^-) = \lim_{h\to 0}\frac{\sqrt{9-(3-h)^2}-0}{-h}$$

$$= \lim_{h\to 0}\frac{\sqrt{6h-h^2}}{-h} = -\infty$$

$$f'(-3^+) = \lim_{h\to 0}\frac{\sqrt{9-(-3+h)^2}-0}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{6h-h^2}}{h} = \infty$$

Thus, f is differentiable on the open interval (–3, 3).

Note: The nth-root function $f(x) = \sqrt[n]{x}$ is non-differentiable at x = 0.

We may combine this result with the previous theorem. Then we see that a root of a differentiable function is differentiable except possibly at the points where the given function is zero. That is, the composition

$$h(x) = \sqrt[n]{g(x)} = [g(x)]^{1/n}$$

of $f(x) = \sqrt[n]{x}$ and the differentiable function g(x) is differentiable at x = a if $g(a) \neq 0$.

Example 4. Show that the function

$$f(x) = \left(\frac{x+1}{x^2+2x+2}\right)^{2/3}$$

is differentiable everywhere except at one point.

Solution Note first that the denominator

$x^2 + 2x + 2 = (x + 1)^2 + 1$ is never zero.

Hence the rational function

$$r(x) = \frac{x+1}{x^2+2x+2}$$

is defined and differentiable everywhere. It then follows from the theorem and the differentiability of the cube root function that $f(x) = [r(x)]^{2/3} = \sqrt[3]{[r(x)]^2}$ is differentiable everywhere except when r(x) = 0.

Here r(x) = 0 at x = – 1.

$$f'(-1^-) = \lim_{h\to 0}\frac{\left(\frac{-h}{h^2+1}\right)^{2/3}-0}{-h} = -\infty$$

$$f'(-1^+) = \lim_{h\to 0}\frac{\left(\frac{h}{h^2+1}\right)^{2/3}-0}{h} = \infty$$

Hence, f is differentiable everywhere except at x = – 1.

Example 5. Let $f(x) = [\sin x] + [\cos x]$, $x \in [0, 2\pi]$, where [.] denotes the greatest integer function.

Find the number of points where f(x) is non-differentiable.

Solution [sin x] is non-differentiable at $x = \dfrac{\pi}{2}, \pi,$ 2π and [cos x] is non-differentiable at $x = 0, \dfrac{\pi}{2}, \dfrac{3\pi}{2}, 2\pi$.

Thus, f(x) is definitely non-differentiable at

$$x = 0, \pi, \frac{3\pi}{2}.$$

We need to check at $x = \frac{\pi}{2}$ and 2π, since both the functions are non-differentiable at these points.

$$f\left(\frac{\pi}{2}\right) = 1,\ f\left(\frac{\pi}{2} - 0\right) = 0$$

and, $f(2\pi) = 1$, $f(2\pi - 0) = -1$.

Thus, f(x) is discontinuous at $x = \frac{\pi}{2}$ and 2π and hence non-differentiable at these points.

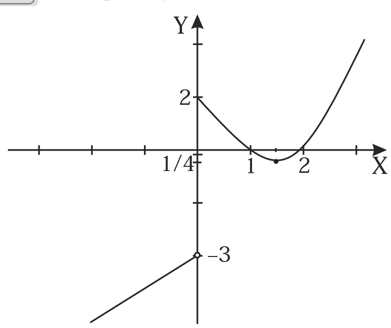
Finally, f is non-differentiable at $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.

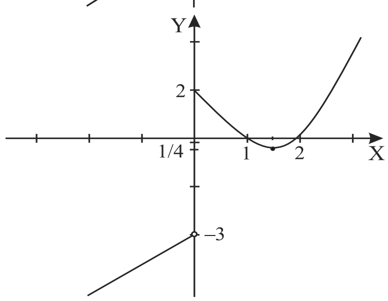
i.e. at 5 points.

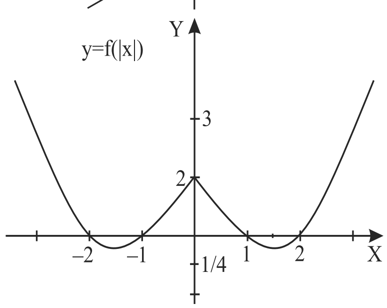
Example 6. If $f(x) = \begin{cases} x - 3 & x < 0 \\ x^2 - 3x + 2 & x \geq 0 \end{cases}$

and $g(x) = f(|x|) + |f(x)|$, then comment on the continuity and differentiability of g(x) by drawing the graph of f(|x|) and, |f(x)|.

Solution Graph of y = f(x)







Continuity of g(x) :

g(x) is continuous at all those points where both f(|x|) and |f(x)| are continuous.

At x = 0, f(|x|) is continuous but |f(x)| is discontinuous.

$\therefore$ g(x) is discontinuous at x = 0 and hence non-differentiable.

At x = 1, f(|x|) is differentiable but |f(x)| is non-differentiable because of corner. Hence, g(x) is non-differentiable at x = 1. Similarly g(x) is non differentiable at x = 2.

Finally, g(x) is discontinuous at x = 0 and it is non differentiable at x = 0, 1, 2.

Theorem Let f(x) and g(x) be defined on an open interval containing the point x = a where f is differentiable at a, f(a) = 0 and g is continuous at a, then the product f.g is differentiable at a.

$$(fg)'(a) = \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h}$$

$$= \lim_{h\to 0} \frac{f(a+h)g(a+h) - 0.g(a)}{h} \quad [\text{since } f(a) = 0]$$

$$= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.g(a+h)$$

$$= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.\lim_{h\to 0} g(a+h) = f'(a).\ g(a)$$

As h approaches zero, g(a+h) approaches g(a) because g is continuous at a.

Therefore, f(x). g(x) is differentiable at x = a.

For example, $f(x) = \sin x.(|x - \pi| + 1)$ is differentiable at $x = \pi$, since sin x is differentiable at $x = \pi$, $\sin \pi = 0$ and $(|x - \pi| + 1)$ is continuous at $x = \pi$.

Corollary:

If f(x) is differentiable at x = a, then the product f(x).|f(x)| is also differentiable at x = a.

For example, $f(x) = \sin x|\sin x|$ is differentiable at x = 0, since sin x is differentiable at x = 0.

Example 7. Let $f(x) = e^{(x-1)} - ax^2 + b$ and

$g(x) = \begin{cases} e^{x-1}, & x \leq 1 \\ x^2, & x > 1 \end{cases}$ with $f'(1) = 2$. Find the values of a and b so that the function $h(x) = f(x) \cdot g(x)$, is differentiable at x = 1.

Solution Note that g is continuous at x = 1, but g ′(1) does not exist.

Also $f'(x) = e^{x-1} - 2ax$

$$f'(1) = 1 - 2a = 2 \Rightarrow a = -\frac{1}{2}$$

In such a situation, for h (x) = f (x)· g (x) to be differentiable at x = 1, f(1) should be 0, using the theorem given above.

$$f(1) = 0 \Rightarrow 1 - a + b = 0$$

$$\Rightarrow \quad b = -\frac{3}{2}.$$

Concept Problems F

1. If it reasonable to assert that the sum F(x) = f(x) + g(x) has no derivative at the point $x = x_0$ if :
   (a) The function f(x) has a derivative at the point x_0, and the function g(x) has no derivative at this point?
   (b) Neither function has a derivative at the point x_0?
2. Suppose that $f(x) = x^2$ and $g(x) = |x|$. Then the composite functions $(f \text{ o } g)(x) = |x|^2 = x^2$ and $(g \text{ o } f)(x) = |x^2| = x^2$ are both differentiable at x = 0 even though g itself is not differentiable at x = 0. Does this contradict the chain rule? Explain.
3. Consider the functions :
   (i) f(x) = x, g(x) = |x|;
   (ii) f(x) = |x|, g(x) = |x|. Is it reasonable to assert that the product F(x) = f(x) g(x) has no derivative at the point $x = x_0$ if :
   (a) The function f(x) has a derivative at the point x_0, and the function g(x) has no derivative at this point ?
   (b) Neither function has a derivative at the point x_0?
4. Find the one-sided derivatives of the function $f(x) = |x - x_0|\, g(x)$ at the point x_0, where g(x) is a function continuous at the point x_0. Does the function f(x) possess a derivative at the point x_0?
5. If f(x) is differentiable at x = a, then prove that F(x) = (f(x) – f(a)).|x – a| is also differentiable at x = a.

Practice Problems E

6. If $f(x) = \tan \pi x$ and $g(x) = |x - 1|$. Is the product function F(x) = f(x) . g(x) differentiable at x = 1 ?
7. If $f(x) = \begin{cases} x^3 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ and $g(x) = \frac{1}{x}$.
   Is the product function F(x) = f(x) . g(x) differentiable for all real x ?
8. Let g(x) be derivable on R and $g(x_0) \neq 0$.
   (a) Show that if $f(x) = (x - x_0)^2 g(x)$, then $f(x_0) = f'(x_0) = 0$ and $f''(x_0) \neq 0$.
   (b What can you say about $f(x) = (x - x_0)^3 g(x)$?
   (c) Generalize to $f(x) = (x - x_0)^n g(x)$.
9. Suppose u = g(x) is differentiable at x = – 5, y = f(u) is differentiable at u = g(–5), and (f o g)′ (–5) is negative. What, if anything, can be said about the values of g ′(–5) and f ′(g(–5))?
10. Let $f(x) = \begin{cases} 1 - x^2 & , x < 0 \\ 2x + 1 & , 0 \leq x < 1 \\ x^2 + 2 & , x \geq 1 \end{cases}$ without finding (fof) (x) explicitly, find whether fof is differentiable at x = 0.
11. Let $f(x) = \begin{cases} 1 - x & 0 \leq x \leq 1 \\ x + 2 & 1 < x < 2 \\ 4 - x & 2 \leq x \leq 4 \end{cases}$, Discuss the continuity and differentiability of fof
12. Let $f(x) = \begin{cases} x - 1, & -1 \leq x < 0 \\ x^2, & 0 \leq x \leq 1 \end{cases}$, Discuss the continuity and differentiability of $h(x) = f(|\sin x|) + |f(\sin x)|$ in $[0, 2\pi]$.
13. Let $f(x) = x^p \dfrac{e^{a/x} - e^{-a/x}}{e^{a/x} + e^{-a/x}}$ when $x \neq 0$ and f(0) = 0, with p, a > 0. Find all possible values of p so that f(x) is differentiable at x = 0.
14. Discuss the differentiability of the function $f(x) = (x^2 - a)\,|x^2 - 5x + 6| + (\sin x)\,|\sin x|$ for all $a \in R$.

3.10 FUNCTIONAL EQUATIONS

We follow the following steps to determine the functions which are differentiable(or which can be proved to be differentiable) and satisfying a given functional rule :

(i) First we write down the expression for $f'(x)$:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

(ii) We manipulate $f(x+h) - f(x)$ in such a way that the given functional rule is applicable. Now we apply the functional rule and simplify the R.H.S. to get $f'(x)$ as a function of x.

(iii) Then we integrate $f'(x)$ to get $f(x)$ as a function of x and a constant of integration. In some cases a differential equation in formed which can be solved to get $f(x)$.

(iv) Finally we apply the boundary conditions to determine the value of the constant of integration.

Example 1. If $f(x+y) = f(x)\cdot f(y), \forall\ x, y \in R$ and $f(x)$ is a differentiable function, then find $f(x)$.

Solution $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0}\frac{f(x)\cdot f(h)-f(x)}{h}$$

$$= \lim_{h\to 0} f(x)\frac{[f(h)-1]}{h} \qquad ...(1)$$

Put $x = 0, y = 0; f(0) = f^2(0)$

$\Rightarrow$ $f(0) = 0$ or $f(0) = 1$

If $f(0) = 0$ then put $y = 0$ in the rule

$f(x) = f(x)\cdot f(0) = 0$

$\Rightarrow$ $f(x) = 0. f(x)$

$\Rightarrow$ $f(x) = 0.$

If $f(0) = 1$, then we proceed from (1) as follows :

$$f'(x) = f(x)\frac{[f(h)-f(0)]}{h} = f(x)\cdot f'(0)$$

[Let $k = f'(0)$]

$\Rightarrow$ $\frac{f'(x)}{f(x)} = k$

$\Rightarrow$ $ln|f(x)| = kx + c$

$|f(x)| = e^{kx+c} = e^{kx}\cdot e^c$

$f(x) = \pm e^{kx}\cdot e^c$

$f(x) = a.e^{kx}$ where $a = \pm e^c$

If $x = 0, f(0) = 1 \Rightarrow a = 1$

$\Rightarrow f(x) = e^{kx}$ where $k = f'(0)$.

Hence, the functions are $f(x) = 0$ and $f(x) = e^{kx}$.

Example 2. If $f(x+y) = f(x) + f(y)\ \forall\ x, y \in R$ and $f(x)$ is a differentiable function, then prove that $f(kx) = k f(x)$ for $\forall\ k, x \in R$.

Solution Given $f(x+y) = f(x) + f(y)$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x)+f(h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = f'(0). \quad [\text{as } f(0) = 0]$$

$\Rightarrow$ $f'(x) = f'(0)$

$\therefore$ $f(x) = f'(0)x + c$

If $x = 0,\ f(0) = 0 \Rightarrow c = 0$

$\therefore$ $f(x) = f'(0)x$

$f(x) = ax$, where $a = f'(0)$

$f(kx) = a\,kx = k\,ax = kf(x)$

Hence, $f(kx) = kf(x)$.

Example 3. A function $f : (0, \infty) \to R$ satisfies the equation $f(x/y) = f(x) - f(y)$. If $f(x)$ is differentiable on $(0, \infty)$ and $\lim_{x\to 0}\frac{f(1+x)}{x} = 3$, then determine $f(x)$.

Solution We have $f(x/y) = f(x) - f(y)$ ...(1)

Putting $x = 1$ and $y = 1$ in equation (1), we have

$f(1) = f(1) - f(1) \Rightarrow f(1) = 0.$

We have $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0}\frac{f\left(\frac{x+h}{x}\right)}{h} \quad [\text{from (1)}]$$

$$= \lim_{h\to 0}\frac{1}{x}\cdot\frac{f\left(1+\frac{h}{x}\right)}{\frac{h}{x}} = \frac{1}{x}\cdot\lim_{\delta\to 0}\frac{f(1+\delta)}{\delta}$$

$$\left[\because \frac{h}{x}\to 0 \text{ as } x\to 0\ \forall\ x\in(0,\infty)\right]$$

$$= \frac{3}{x} \qquad \left[\text{from } \lim_{x\to 0}\frac{f(1+x)}{x} = 3\right]$$

To find f(x), write the above equation as

$$\frac{df}{dx} = \frac{3}{x}$$

On integrating both sides w.r.t. x we get

$f(x) = 3 \ln x + c$, where c is a constant.

Now, using the condition $f(1) = 0$, we have

$f(1) = c$ which gives $c = 0$

Hence, we have $f(x) = 3 \ln x$.

Example 4. A differentiable function satisfies the relation $f(x+y) = f(x) + f(y) + 2xy - 1 \ \forall\, x, y \in R$. If $f'(0) = \sqrt{3 + a - a^2}$, find f(x) and prove that $f(x) > 0 \ \forall\, x \in R$.

Solution $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{f(x) + f(h) + 2xh - 1 - f(x)}{h}$$

$$= 2x + \lim_{h\to 0} \frac{f(h) - 1}{h} \quad [\text{Put } x = 0, y = 0 \text{ to get } f(0) = 1]$$

$$= 2x + \lim_{h\to 0} \frac{f(h) - f(0)}{h}$$

$\Rightarrow \quad f'(x) = 2x + f'(0)$

Integrating, $f(x) = x^2 + f'(0)x + c$

If $x = 0$; $f(0) = 1 \Rightarrow c = 1$

$\therefore \quad f(x) = x^2 + \left(\sqrt{3 + a - a^2}\right)x + 1$

Now $D = 3 + a - a^2 - 4 = -(a^2 - a + 1) < 0$

$\Rightarrow \quad f(x) > 0 \ \forall\, x \in R$.

Example 5. Given a function g which has derivative g′(x) for all x satisfying $g'(0) = 2$ and $g(x+y) = e^y g(x) + e^x g(y)$ for all $x, y \in R$, $g(5) = 32$. Find the value of $g'(5) - 2e^5$.

Solution Putting $x = y = 0$ in

$g(x+y) = e^y g(x) + e^x g(y)$ we get $g(0) = 2g(0)$

$\Rightarrow \quad g(0) = 0$

So $\quad 2 = g'(0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{g(h)}{h}$

Also $g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$

$$= \lim_{h\to 0} \frac{e^x g(h) + e^h g(x) - g(x)}{h}$$

$$= \lim_{h\to 0} \left(e^x \frac{g(h)}{h} + \frac{e^h - 1}{h} g(x) \right)$$

$$= e^x \lim_{h\to 0} \frac{g(h)}{h} + 1.\, g(x)$$

$$= g(x) + 2e^x.$$

Thus, $g'(5) - 2e^5 = g(5) = 32$.

Example 6. A differentiable function f(x) satisfies the condition $f(x+y) = f(x) + f(y) + xy$ for all $x, y \in R$ and

$\lim_{h\to 0} \frac{1}{h} f(h) = 3$ then find the least value of f(x).

Solution $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{f(x) + f(h) + hx - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(h)}{h} + x = 3 + x$$

Integrating $f(x) = 3x + \frac{x^2}{2} + k \qquad \text{...(1)}$

Putting x = 0, y = 0 in the given relation

$f(0) = f(0) + f(0) + 0 \Rightarrow f(0) = 0$

Now from (1) we have $f(0) = 0 + k \Rightarrow k = 0$

$\therefore \quad f(x) = 3x + \frac{x^2}{2}$.

It is a quadratic function whose least value occurs at $x = -3$. The least value is $-\frac{9}{2}$.

Example 7. Let $f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}$ for all real x and y. If $f'(0) = -1$ and $f(0) = 1$, then find f(2).

Solution Given $f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}$

Replacing x by 2x and y by 0, then $f(x) = \frac{f(2x) + f(0)}{2}$

$\Rightarrow \quad f(2x) + f(0) = 2f(x) \Rightarrow f(2x) - 2f(x) = -f(0) \quad \text{...(1)}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f\left(\frac{2x+2h}{2}\right) - f(x)}{h}$$

$$= \lim_{h\to 0} \left\{ \frac{\frac{f(2x) + f(2h)}{2} - f(x)}{h} \right\}$$

$$= \lim_{h\to 0}\left\{\frac{f(2x)+f(2h)-2f(x)}{2h}\right\}$$

$$= \lim_{h\to 0}\left\{\frac{f(2h)-f(0)}{2h}\right\} \quad \text{(from (1))}$$

$= f'(0)$

$= -1\ \forall\ x \in R$ (given)

Integrating, we get $f(x) = -x + c$

Putting $x = 0$, then $f(0) = 0 + c = 1$ (given)

$\therefore$ $c = 1$ then $f(x) = 1 - x$

$\therefore$ $f(2) = 1 - 2 = -1$.

Alternative 1 :

$\because$ $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$

Differentiating both sides w.r.t x treating y as constant.

$\therefore$ $f'\left(\frac{x+y}{2}\right)\cdot\frac{1}{2} = \frac{f'(x)+0}{2} \Rightarrow f'\left(\frac{x+y}{2}\right) = f'(x).$

Replacing x by 0 and y by 2x,

then $f'(x) = f'(0) = -1$ (given)

Integrating, we have $f(x) = -x + c$.

Putting $x = 0$, $f(x) = 0 + c = 1$ (given)

$\therefore$ $c = 1$

Hence, $f(x) = -x + 1$ then $f(2) = -2 + 1 = -1$.

Alternative 2 :

Suppose $A(x, f(x))$ and $B(y, f(y))$ be any two points on the curve $y = f(x)$.

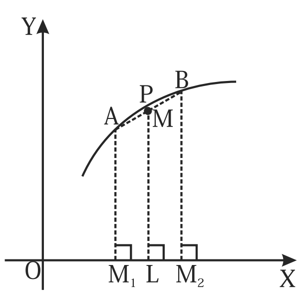
If M is the mid point of AB, then coordinates of M are

$$\left(\frac{x+y}{2}, \frac{f(x)+f(y)}{2}\right)$$

According to the graph, coordinates of P are

$$\left(\frac{x+y}{2}, f\left(\frac{x+y}{2}\right)\right)$$

and $PL > ML \Rightarrow \left(\frac{x+y}{2}\right) > \frac{f(x)+f(y)}{2}$



But given $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ which is possible when $P \rightarrow M$, i.e., P lies on AB. Hence, $y = f(x)$ must be a linear function.

Let $f(x) = ax + b \Rightarrow f(0) = 0 + b = 1$ (given)

and $f'(x) = a \Rightarrow f'(0) = a = -1$ (given)

$\therefore$ $f(x) = -x + 1$

$\therefore$ $f(2) = -2 + 1 = -1$.

Example 8. If $f\left(\frac{x+y}{3}\right) = \frac{2+f(x)+f(y)}{3}$ for all real x and y and $f'(2) = 2$ then determine $y = f(x)$.

Solution $f\left(\frac{x+y}{3}\right) = \frac{2+f(x)+f(y)}{3}$ ...(1)

Replacing x by 3x and y by 0 then

$$f(x) = \frac{2+f(3x)+f(0)}{3}$$

$\Rightarrow$ $f(3x) - 3f(x) + 2 = -f(0)$ ...(2)

Putting $x = 0$ and $y = 0$ in (1),

we get $f(0) = 2$ ...(3)

Now, $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0}\frac{f\left(\frac{3x+3h}{3}\right)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{2+f(3x)+f(3h)}{3}-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(3x)-3f(x)+f(3h)+2}{3h}$$

$$= \lim_{h\to 0}\frac{f(3h)-f(0)}{3h} \quad \text{\{from 2)\}}$$

$= f'(0) = c$ (say)

$\therefore$ $f'(x) = c$

At $x = 2$, $f'(2) = c = 2$ (given)

$\therefore$ $f'(x) = 2$

Integrating both sides, we get $f(x) = 2x + a$

Putting $x = 0$ then

$f(0) = 0 + a = 2$ {from (2)}

$\therefore$ $a = 2$

then $f(x) = 2x + 2$.

Alternative :

We have $f\left(\frac{x+y}{3}\right) = \frac{2+f(x)+f(y)}{3}$

Differentiating both sides w.r.t. x treating y as constant,

we get $f'\left(\frac{x+y}{3}\right)\left(\frac{1}{3}\right)=\frac{2+f'(x)+0}{3}$

Now replacing x by 0 and y by 3x, then

$f'(x)=f'(0)=c$ (say)

At $x=2$,

$f'(2)=c=2$ (given)

$\therefore\quad f'(x)=2$

On integrating we get

$f(x)=2x+a$

Putting $x=0$, then $f(0)=0+a=2$ {from (1)}

$\therefore\quad f(x)=2x+2.$

Example 9. Let $f(xy)=xf(y)+yf(x)$ for all $x, y\in R^+$ and f(x) be differentiable in $(0,\infty)$ then determine f(x).

Solution Given $f(xy)=xf(y)+yf(x)$

Replacing x by 1 and y by x then we get $x\,f(1)=0$

$\therefore\quad f(1)=0, x\neq 0 \qquad (\because x, y, \in R^+)$

$$\text{Now, } f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f\left(x\left(1+\frac{h}{x}\right)\right)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{xf\left(1+\frac{h}{x}\right)+\left(1+\frac{h}{x}\right)f(x)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{xf\left(1+\frac{h}{x}\right)+\frac{h}{x}f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)}{\left(\frac{h}{x}\right)}+\lim_{h\to 0}\frac{f(x)}{x}$$

$$=f'(1)+\frac{f(x)}{x}$$

$$\Rightarrow\quad f'(x)-\frac{f(x)}{x}=f'(1)$$

$$\Rightarrow\quad \frac{xf'(x)-f(x)}{x^2}=\frac{f'(1)}{x}$$

$$\Rightarrow\quad \frac{f(x)}{x}\left\{\frac{f(x)}{x}\right\}=\frac{f'(1)}{x}$$

On integrating w.r.t.x and taking limit 1 to x,

we have $\frac{f(x)}{x}-\frac{f(1)}{1}=f'(1)(\ln x-\ln 1)$

$$\Rightarrow\quad \frac{f(x)}{x}-0=f'(1)\ln x \qquad (\because f(1)=0)$$

$\therefore\quad f(x)=f'(1)(x\ln x).$

***Alternative* :**

Given $f(xy)=xf(y)+yf(x)$

Differentiating both sides w.r.t. x treating y as constant,

$f'(xy)\,.\,y=f(y)+y\,f'(x)$

Putting $y=x$ and $x=1$, then

$f'(xy).\,x=f(x)+xf'(x)$

$$\Rightarrow\quad \frac{xf'(x)-f(x)}{x^2}=\frac{f'(1)}{x}$$

$$\Rightarrow\quad \frac{d}{dx}\left(\frac{f(x)}{x}\right)=\frac{f'(1)}{x}$$

Integrating both sides w.r.t. x taking limit 1 to x,

$$\frac{f(x)}{x}-\frac{f(1)}{1}=f'(1)\{\ln x-\ln 1\}$$

$$\Rightarrow\quad \frac{f(x)}{x}-0=f'(1)\ln x \qquad (\because f(1)=0)$$

Hence, $f(x)=f'(1)(x\ln x)$.

Example 10. If $f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right)$ for all $x, y\in R\,(xy<1)$ and $\lim_{x\to 0}\frac{f(x)}{x}=2$. Find $f\left(\frac{1}{\sqrt{3}}\right)$ and $f'(1)$.

Solution $f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right)$ ...(1)

Putting $x=y=0$, we get $f(0)=0$.

Putting $y=-x$, we get $f(x)+f(-x)=f(0)$

$\Rightarrow\quad f(-x)=-f(x)$ ...(2)

Also, $\lim_{x\to 0}\frac{f(x)}{x}=2$

$$\text{Now, }(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad ...(3)$$

$$=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad \text{(using (2))}$$

$$\Rightarrow\quad f'(x)=\lim_{h\to 0}\frac{\left(\frac{x+h-x}{1-(x+h)(-x)}\right)}{h} \qquad \text{(using (1))}$$

$$\Rightarrow \quad f'(x)=\lim_{h\to 0}\left[\frac{f\left(\frac{h}{1+x(x+h)}\right)}{h}\right]$$

$$\Rightarrow \quad f'(x)=\lim_{h\to 0}\frac{f\left(\frac{h}{1+xh+x^2}\right)}{\left(\frac{h}{1+xh+x^2}\right)}\times\left(\frac{1}{1+xh+x^2}\right)$$

$$\Rightarrow \quad f'(x)=\lim_{h\to 0}\frac{f\left(\frac{h}{1+xh+x^2}\right)}{\left(\frac{h}{1+xh+x^2}\right)}\times\lim_{h\to 0}\frac{1}{1+xh+x^2}$$

$$\left(\text{using }\lim_{x\to 0}\frac{f(x)}{x}=2\right)$$

$$\Rightarrow \quad f'(x)=2\times\frac{1}{1+x^2} \qquad \Rightarrow f'(x)=\frac{2}{1+x^2}$$

Integrating both sides, we get

$f(x)=2\tan^{-1}(x)+ck$, where $f(0)=0\Rightarrow c=0$

Thus, $f(x)=2\tan^{-1}x$.

Hence, $f\left(\frac{1}{\sqrt{3}}\right)=2\,\frac{\pi}{6}=\frac{\pi}{3}$, and

$f'(1)=\frac{2}{1+1^2}=\frac{2}{2}=1$.

Example 11. If $e^{-xy}f(xy)=e^{-x}f(x)+e^{-y}f(y)\ \forall x,$ $y\in R^+$, and $f'(1)=e$, determine $f(x)$.

Solution Given $e^{-xy}f(xy)=e^{-x}f(x)\,e^{-y}f(y)$ ...(1)

Putting $x=y=1$ in (1), we get, $f(1)=0$ ...(2)

$$\text{Now, } f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f\left(x\left(1+\frac{h}{x}\right)\right)-f(x.1)}{h}$$

$$=\lim_{h\to 0}\frac{e^{x+h}\left\{e^{-x}f(x)+e^{-1-\frac{h}{x}}f\left(1+\frac{h}{x}\right)\right\}-2^x(e^{-x}f(x)+e^{-1}f(1))}{h}$$

$$=\lim_{h\to 0}\frac{e^hf(x)+e^{x+h-1-\frac{h}{x}}f\left(1+\frac{h}{x}\right)-f(x)-e^{x-1}f(1)}{h}$$

$$=f(x)\lim_{h\to 0}\left(\frac{e^h-1}{h}\right)+e^{(x-1)}\lim_{h\to 0}\frac{e^{h-\frac{h}{x}}f\left(1+\frac{h}{x}\right)}{x.\frac{h}{x}}$$

$(\because f(1)=0)$

$$=f(x).1+e^{x-1}.\frac{f'(1)}{x}$$

$$=f(x)+\frac{e^{x-1}.e}{x} \qquad (\because f'(1)=e)$$

$$f'(x)=f(x)+\frac{e^x}{x} \Rightarrow e^{-x}f'(x)-e^{-x}f(x)=\frac{1}{x}$$

$$\Rightarrow \frac{d}{dx}(e^{-x}f(x))=\frac{1}{x}$$

On integrating we have

$e^{-x}f(x)=\ln x+c$ at $x=1$, $c=0$

$\therefore$ $f(x)=e^x\ln x$.

Example 12. A function $f:(-1,1)\to\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$ satisfies the equation

$$f(x)+f(y)=f(x\sqrt{1-y^2}+y\sqrt{1-x^2}).$$

(i) Show that f(x) is odd.

(ii) If f(x) is differentiable on (–1, 1) and $f'(0)=1$, then show that $f'(x)=\frac{1}{\sqrt{1-x^2}}$

(iii) Hence, determine f(x).

Solution

(i) $f(x)+f(y)=f(x\sqrt{1-y^2}+y\sqrt{1-x^2})$. ...(1)

Putting $y=-x$ in equation (1), we have

$f(x)+f(-x)=f(x\sqrt{1-x^2}-x\sqrt{1-x^2})$

$\Rightarrow$ $f(x)+f(-x)=f(0)$

Putting $x=0$ and $y=0$ in equation (1), we have

$f(0)+f(0)=f(0)\Rightarrow f(0)=0$.

Hence, $f(x)+f(-x)=0$ ...(2)

Thus f(x) is odd.

(ii) Now, $f'(0)=1$

$$\Rightarrow \lim_{\delta\to 0}\frac{f(\delta)-f(0)}{\delta}=1 \qquad [\text{using } f(0)=0]$$

$$\Rightarrow \lim_{\delta\to 0}\frac{f(\delta)}{\delta}=1$$

$$\text{Now } f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)+f(-x)}{h} \qquad [\text{using}(2)]$$

$$=\lim_{h\to 0}\frac{f\{x+h)\sqrt{1-x^2}-x\sqrt{1-(x+h)^2}\}}{h}$$

[using (1)]

$$= \lim_{\delta\to 0}\frac{f(\delta)}{\delta}.\lim_{h\to 0}\frac{(x+h)\sqrt{1-x^2}-x\sqrt{1-(x+h)^2}}{h}$$

assuming $\delta = (x+h)\sqrt{1-x^2}-x\sqrt{1-(x+h)^2}$

$$= \lim_{h\to 0}\frac{(x+h)\sqrt{1-x^2}-x\sqrt{1-(x+h)^2}}{h}$$

$$= \lim_{h\to 0}\frac{x(\sqrt{1-x^2}-x\sqrt{1-(x+h)^2})}{h}+\frac{h\sqrt{1-x^2}}{h}$$

$$= \sqrt{1-x^2}+\lim_{h\to 0}\frac{x(1-x^2-1+(x+h)^2)}{h(\sqrt{1-x^2}+\sqrt{1-(x+h)^2})}$$

$$= \sqrt{1-x^2}+\frac{x^2}{\sqrt{1-x^2}}=\frac{1}{\sqrt{1-x^2}}$$

Hence, $f'(x)=\dfrac{1}{\sqrt{1-x^2}}$ ...(3)

(iii) Integrating both sides of (3) gives $f(x) = \sin^{-1}x + c$ (c is a constant). Using the condition $f(0) = 0$ gives $c = 0$, and hence we have $f(x) = \sin^{-1}x$.

Example 13. A differentiable function f satisfies the relation $f(x+y) - 2f(x-y) + f(x) - 2f(y) = y - 2$ $\forall\, x, y \in R$. Find $f(x)$.

Solution $f(x+y) - 2f(x-y) + f(x) - 2f(y) = y - 2$ ...(1)

Put $x = y = 0$ in (1)

$\Rightarrow f(0) = 1$

Put $y = h$ in (1)

$f(x+h) - 2f(x-h) + f(x) - 2f(x) = h - 2$

$\Rightarrow f(x+h) - f(x) - 2[(f(x-h) - f(x)] - 2(f(h) - 1) = h$

Dividing by h on both sides and applying limit $h\to 0$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\frac{2\left(f(x-h)-f(x)\right)}{-h}-\left(\frac{2f(h)-1}{h}\right)=1$$

$\Rightarrow f'(x) + 2f'(x) - 2f'(0) = 1$ [$\because$ f is differentiable]

$\Rightarrow 3f'(x) = 1 + 2f'(0)$ ...(2)

Putting $x = 0$ in (2) we get $f'(0) = 1$.

$\Rightarrow f'(x) = 1$

$\therefore f(x) = x + c$

Since $f(0) = 1$, $c = 1$, we have $f(x) = x + 1$.

Example 14. A differentiable function f satisfies $f(x + y) + f(x - y) - (y + 2) f(x) + y(x^2 - 2y) = 0$ $\forall$ x, $y \in R$. Find $f(x)$.

Solution Put $y = h$

$f(x+h) - f(x) + f(x-h) - f(x) - hf(x) + h(x^2 - 2h) = 0$

Dividing by h on both sides

$$\frac{f(x+h)-f(x)}{h}+\frac{f(x-h)-f(x)}{h}-f(x)+x^2-2h=0$$

and applying limit $h\to 0$ on both sides, we get

$f'(x) - f'(x) - f(x) + x^2 = 0$

$\Rightarrow f(x) = x^2$.

Example 15. A twice differentiable function f satisfies the relation $f(x^2 + y^2) = f(x^2 - y^2) + f(2xy)$ $\forall$ x, $y \in R$. If $f(0) = 0$ and $f''(0) = 2$, find $f(x)$.

Solution $f(x^2 + y^2) = f(x^2 - y^2) + f(2xy)$

Put $y = h$

$f(x^2 + h^2) = f(x^2 - h^2) + f(2xh)$

Dividing by h^2 on both sides

$$\frac{f(x^2+h^2)-f(x^2)}{h^2}=\frac{f(x^2-h^2)-f(x^2)}{h^2}+\frac{f(2xh)}{h^2} \quad ...(1)$$

Now $\displaystyle\lim_{h\to 0}\frac{f(2xh)}{h^2}=\lim_{h\to 0}\frac{f'(2xh).2x}{2h}$ (using L'Hospital's rule)

$$= \lim_{h\to 0}\frac{xf''(2xh).2x}{1}$$

$= 2x^2f''(0)$.

Applying limit $h \to 0$ on both sides of (1), we get

$f'(x^2) = -f'(x^2) + 2x^2f''(0)$

$2f'(x^2) = 2x^2f''(0)$

Replacing x^2 by t

$2f'(t) = 2t\,f''(0) \Rightarrow f'(t) = 2t$

Integrating both sides w.r.t. t

$\int f'(t)dt = \int 2t\,dt$

$f(t) = t^2 + c$

Since $f(0) = 0$, $c = 0$.

$\therefore f(t) = t^2$ or, $f(x) = x^2$.

Example 16. f(x) is a differentiable function satisfy the relationship $f^2(x) + f^2(y) + 2(xy - 1) = f^2(x + y)$

$\forall$ x, y $\in$ R. Also $f(x) > 0$ $\forall$ $x \in R$, and $f(\sqrt{2}) = 2$. Determine f(x).

Solution Put $x = 0$ and $y = 0$ $\Rightarrow$ $f^2(0) = 2$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f^2(x+h) - f^2(x)}{[f(x+h) + f(x)].h}$$

$$= \lim_{h\to 0} \frac{f^2(h) + 2(xh - 1)}{2f(x)h}$$

$$= \frac{1}{2f(x)} \lim_{h\to 0} \left[\frac{2xh}{h} + \frac{f^2(h) - 2}{h}\right]$$

$$= \frac{1}{2f(x)} \left[2x + \lim_{h\to 0} \frac{f^2(h) - f^2(0)}{h}\right]$$

$$= \frac{1}{2f(x)} \left[2x + \lim_{h\to 0} \frac{f(h) - f(0)}{h}.(f(h) + f(0))\right]$$

$$f'(x) = \frac{1}{2f(x)}[2x + 2f(0).f'(0)]$$

$\therefore$ $f(x).f'(x) = x + f(0).f'(0)$

$\Rightarrow$ $f(x).f'(x) = x + \lambda$, where $\lambda = f(0)\cdot f'(0)$

Integrating both sides,

$$\frac{f^2(x)}{2} = \frac{x^2}{2} + \lambda x + c$$

$f^2(x) = x^2 + 2\lambda x + c$

at $x = 0$, $f^2(0) = 2$ $\Rightarrow$ $c = 2$

at $x = \sqrt{2}$, $f^2(\sqrt{2}) = 4$ $\Rightarrow \lambda = 0$

$\therefore$ $f^2(x) = x^2 + 2$

$\Rightarrow$ $f(x) = \sqrt{x^2 + 2}$ (as $f(x) > 0$)

Example 17. function $f : (0, \infty) \to R$ satisfies the equation $f(xy) = 2f(x) - f\left(\frac{x}{y}\right)$. If f is differentiable on R^+ and $f(1) = 0$, $f'(1) = 1$, then show that

(i) $f(y) = -f\left(\frac{1}{y}\right)$ (ii) $f(x) + f(y) = f\left(\frac{x}{y}\right)$

and hence determine f(x).

Solution We have $f(xy) = 2f(x) - f\left(\frac{x}{y}\right)$ ...(1)

(i) Putting x = 1 in (1), we have

$2f(1) = f(y) - \left(\frac{1}{y}\right)$

$\Rightarrow$ $f(y) = -f\left(\frac{1}{y}\right)$ [since f(1) = 0] ...(2)

(ii) In (1) we exchange x and y to get

$f(xy) = 2f(y) - f\left(\frac{y}{x}\right)$

$\Rightarrow$ $f(xy) = 2f(y) + f\left(\frac{x}{y}\right)$ [using (2)] ...(3)

Now we subtract (1) from (3)

$0 = 2f(y) - 2f(x) + 2f\left(\frac{x}{y}\right)$

$\Rightarrow$ $f(x) - f(y) = f\left(\frac{x}{y}\right)$ ...(4)

We have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f\left(\frac{x+h}{x}\right)}{h} \text{ [using (4)]}$$

$$= \lim_{h\to 0} \frac{f\left(1 + \frac{h}{x}\right)}{x.\frac{h}{x}} = \frac{f'(1)}{x}$$

i.e. $f'(x) = \frac{1}{x}$

This gives $f(x) = \ln x + c$

Now, using the condition f(1) = 0 gives c = 0.

Hence, we have $f(x) = \ln x$.

Example 18. A differentiable function f satisfies the relation $f(x + y) + f(xy - 1) = f(x) + f(y) + f(xy)$ $\forall$ x, y $\in$ R.

If $f(1) = 2$, $f'(0) = 1$ and $f'(-1) = -1$, find f(x).

Solution Putting x = 0, y = 0 in the given rule,

$f(0) + f(-1) = 3f(0)$

$\Rightarrow$ $f(-1) = 2f(0)$ ...(1)

Now, putting y = h

$f(x + h) + f(xh - 1) = f(x) + f(h) + f(xh)$

$\Rightarrow$ $f(x+h)-f(x)+f(-1+xh)-f(-1)$
$= f(h)+f(xh)-f(-1)$

$\Rightarrow$ $f(x+h)-f(x)+f(-1+xh)-f(-1)$
$= f(h)-f(0)+f(xh)-f(0)$ [using(1)]

Dividing both sides by h,

$$\frac{f(x+h)-f(x)}{h}+\frac{f(-1+xh)-f(-1)}{xh}.x$$

$$=\frac{f(h)-f(0)}{h}+\frac{f(xh)-f(0)}{xh}.x$$

Now, applying limit $h \to 0$ on both sides, we get

$f'(x)+xf'(-1)=f'(0)+x\,f'(0)$

Given that $f'(0)=1$, $f'(-1)=-1$, we have

$f'(x)=2x+1$

$f(x)=x^2+x+c$

$f(1)=2 \Rightarrow c=0$

$\therefore$ $f(x)=x^2+x.$

Example 19. Let $f : R \to R$ be such that for all x and y in R, $|f(x)-f(y)| \le |x-y|^3$. Prove that f(x) is a constant function.

Solution We are given that

$|f(x)-f(y)| \le |x-y|^3$ ...(1)

Let x be any real number and let y be chosen arbitrarily close to x but not equal to x. Then writing (1) as

$$\left|\frac{f(x)-f(y)}{x-y}\right| \le |x-y|^2$$

and letting $y \to x$, we get

$$\lim_{y\to x}\left|\frac{f(x)-f(y)}{x-y}\right| \le \lim_{y\to x}|x-y|^2 \quad ...(2)$$

Since $\lim_{y\to x}\frac{f(x)-f(y)}{x-y}=f'(x)$, we see from (2) that

$|f'(x)| \le 0 \Rightarrow f'(x)=0$

$\Rightarrow$ $f(x)=c.$

Hence f(x) is a constant function.

Example 20. Suppose

$p(x)=a_0+a_1x+a_2x^2+...+a_nx^n.$

If $|p(x)| \le |e^{x-1}-1|$ for all $x \ge 0$, prove that $|a_1+2a_2+...+na_n| \le 1.$

Solution Given $p(x)=a_0+a_1x+a_2x^2+...+a_nx^n$

$\therefore$ $p'(x)=0+a_1+2a_2x+...+na_nx^{n-1}$

$\Rightarrow$ $p'(1)=a_1+2a_2+...+na_n$

Now, $|p(1)| \le |e^{1-1}-1|=|e^0-1|=0$

$\Rightarrow$ $|p(1)| \le 0 \Rightarrow p(1)=0$

As $|p(x)| \le |e^{x-1}-1|$

we get $|p(1+h)| \le |e^h-1|\ \forall\ h>-1, h \ne 0$

$\Rightarrow$ $|p(1+h)-p(1)| \le |e^h-1|$ $(\because p(1)=0)$

$$\Rightarrow \left|\frac{p(1+h)-p(1)}{h}\right| \le \left|\frac{p^h-1}{h}\right|$$

Taking limit as $h \to 0$ on both sides, then

$$\Rightarrow \lim_{h\to 0}\left|\frac{p(1+h)-p(1)}{h}\right| \le \lim_{h\to 0}\left|\frac{e^h-1}{h}\right|$$

$$\Rightarrow \left|\lim_{h\to 0}\frac{p(1+h)-p(1)}{h}\right| \le \left|\lim_{h\to 0}\frac{e^h-1}{h}\right|$$

$\Rightarrow$ $|p'(1)| \le 1$

$\Rightarrow$ $|a_1+2a_2+...+na_n| \le 1$ {from (1)}

Practice Problems F

1. A function $f : R \to R+$ satisfies $f(x+y)=f(x)$. $f(y)\ \forall\ x \in R$. If $f'(0)=2$, then show that $f'(x)=2f(x)$.
2. Let $f(xy)=f(x)\,f(y)\ \forall\ x, y \in R$ and $f(1) \ne 0$, $f'(1)=1$, prove that f is differentiable for all $x \ne 0$. Hence determine f(x).
3. Let $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$ for real x and y. If $f'(0)$ exists and equals -1 and $f(0)=1$ then find the value of f(2)
4. If f(x) satisfies $f(1-x)=f(x)\ \forall\ x \in R$ and $f'(1)=0$ then find $f'(0)$ if it exists.
5. Let $f\left(\frac{x+y}{n}\right)=\frac{f(x)+f(y)}{n}\ \forall\ x, y \in R, n>2$ and $f'(0)=2$ then find f(x).
6. Let $f\left(\frac{xy}{2}\right)=\frac{f(x)f(y)}{6}$ for all real x and y. If $f(1)=f'(1)=3$, then prove that one of the functions f satisfies $f(x)+f(1-x)=3$ for all non-zero real x.
7. If $2f(x)=f(xy)+f\left(\frac{x}{y}\right)\ \forall\ x, y \in R^+$, $f(1)=0$ and $f'(1)=1$ then find $f'(2)$ and f(2).
8. Let $f : R \to R$ such that $f(x+y)$

$= f(x) + f(y) + e^{x+y}(x+y) - xe^x - ye^y + 2xy \ \forall\ x, y \in R$ then find f(x) given that $f'(0) = 1$.

9. If $f\left(\dfrac{2x+3y}{5}\right) = \dfrac{2f(x)+3f(y)}{5}\ \forall\ x, y \in R$, $f(0) = 1$, $f'(0) = -1$, prove that

$$\sum_{r=2}^{n+1}(f(r))^3 = -\frac{n^2(n+1)^2}{4}.$$

10. A differentiable function f satisfies the relation $f(x+y)[f(x) - f(y)] = f(x-y)[f(x) + f(y)]$, where $f(1) = 2$. Find f(x).

Target Exercises for JEE Advanced

Problem 1. Check the differentiability of f(x) at x = 0, where

$$f(x) = x\left(\frac{2 - e^{|x|+\{x\}}}{|x| + \{x\}}\right),\quad x \neq 0$$
$$= \frac{1}{2},\quad x = 0$$

where { } represents the fractional part function.

Solution

$$f'(0^+) = \lim_{h\to 0}\frac{f(0+h) - f(0)}{h} = \lim_{h\to 0}\frac{h\left(\dfrac{2 - e^{|h|+\{h\}}}{|h|+\{h\}}\right) - \dfrac{1}{2}}{h}$$

$$= \lim_{h\to 0}\frac{h\left(\dfrac{2 - e^{2h}}{2h}\right) - \dfrac{1}{2}}{h}$$

$[\because |h| + \{h\} = h + h = 2h$ for small $h > 0]$

$$= \lim_{h\to 0}\frac{1 - e^{2h}}{2h} = -\ln e = -1.$$

$$f'(0^-) = \lim_{h\to 0}\frac{f(0) = f(0-h)}{h}$$

$$= \lim_{h\to 0}\frac{\dfrac{1}{2} - (-h)\left(\dfrac{2 - e^{|-h|+\{-h\}}}{|-h| + \{-h\}}\right)}{h}$$

$$= \lim_{h\to 0}\frac{\dfrac{1}{2} + h(2-e)}{h}$$

$[\because |-h| + \{-h\} = h + 1 - h = 1$ for small $h > 0]$

$$= \lim_{h\to 0}\frac{1}{2h} = \infty$$

Hence, f is non-differentiable at x = 0.

Problem 2. Find $f'(0)$ if $f(x) = h(x)\cos(1/x), x \neq 0$, $f(0) = 0$, where h(x) is an even function differentiable at x = 0 and h(0) = 0.

Solution The existence of $h'(0)$ implies that

$$\lim_{\delta\to 0}\frac{h(\delta) - h(0)}{\delta} = \lim_{\delta\to 0}\frac{h(0) - h(-\delta)}{\delta}$$

$$\Rightarrow \lim_{\delta\to 0}\frac{h(\delta)}{\delta} = \lim_{\delta\to 0}\frac{-h(\delta)}{\delta}$$

$$\Rightarrow 2\lim_{\delta\to 0}\frac{h(\delta)}{\delta} = 0 \qquad ...(1)$$

Now consider $f'(x)$ at x = 0.

$$\text{R.H.D.} = \lim_{\delta\to 0}\frac{f(\delta) - f(0)}{\delta} = \lim_{\delta\to 0}\frac{h(\delta)\cos(1/\delta)}{\delta}$$

$$= \lim_{\delta\to 0}\frac{h(\delta)}{\delta}.\lim_{\delta\to 0}\cos(1/\delta)$$

$= 0$ [using (1)] ...(2)

$$\text{L.H.D.} = \lim_{\delta\to 0}\frac{f(0) - f(-\delta)}{\delta} = \lim_{\delta\to 0}\frac{-h(-\delta)\cos(-1/\delta)}{\delta}$$

$$= \lim_{\delta\to 0}\frac{-h(\delta)}{\delta}.\lim_{\delta\to 0}\cos(1/\delta)$$

$= 0$ [using (1)] ...(3)

Since R.H.D. = L.H.D. = 0, we have $f'(0) = 0$.

Problem 3. If $f(x) = \begin{cases} xe^{-\left(\frac{1}{x} + \frac{1}{|x|}\right)}, & x \neq 0, \\ a, & x = 0 \end{cases}$

find the value of 'a' such that f(x) is differentiable at x = 0.

Solution

$$f'(0) = \lim_{h\to 0}\frac{f(h) - f(0)}{h} = \lim_{h\to 0}\frac{h\ e^{-\left(\frac{1}{h} + \frac{1}{|h|}\right)} - a}{h}$$

$$f'(0^+) = \lim_{h\to 0}\frac{h\ e^{-\left(\frac{1}{h} + \frac{1}{h}\right)} - a}{h}$$

$$= \lim_{h\to 0}\frac{h\,.\,e^{-2/h} - a}{h} = 0 - \lim_{h\to 0}\frac{a}{h} = 0, \text{ provided } a = 0.$$

$$\text{Also } f'(0^-) = \lim_{h\to 0}\frac{-h\,.\,e^{-\left(-\frac{1}{h} + \frac{1}{h}\right)} - a}{-h}$$

$$= 1 + \lim_{h\to 0}\frac{a}{h} = 1, \text{ provided } a = 0.$$

Hence, a = 0.

Problem 4. Let $f(x)=\begin{cases} |x-3|\,[x], & x \ge 1 \\ \sin\left(\frac{\pi x}{2}\right), & x<1 \end{cases}$

where [.] denotes the greatest integer function.
Find whether f(x) is
(i) is continuous at x = 0,
(ii) is differentiable at x = 0,
(iii) is continuous but not differentiable at x = 1,
(iv) is continuous but not differentiable at x = 3/2.

Solution

(i) $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{h \to 0} f(0-h) = \lim\limits_{h \to 0} \sin\left(\frac{-\pi h}{2}\right) = 0$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{h \to 0} f(0+h) = \lim\limits_{h \to 0} \sin\frac{\pi h}{2} = 0$

f(0) = 0

∴ f(x) is continuous at x = 0.

(ii) $f'(0^-) = \lim\limits_{h \to 0} \frac{f(0-h)-f(0)}{-h}$

$= \lim\limits_{h \to 0} \frac{-\sin\frac{\pi h}{2} - 0}{-h} = \lim\limits_{h \to 0} \frac{\pi}{2}\, \frac{\sin\frac{\pi h}{2}}{\frac{\pi h}{2}} = \frac{\pi}{2}$

$f'(0^+) = \lim\limits_{h \to 0} \frac{f(0+h)-f(0)}{h}$

$= \lim\limits_{h \to 0} \frac{\pi}{2}\, \frac{\sin\frac{\pi h}{2}}{\frac{\pi h}{2}} = \frac{\pi}{2}$

$f'(0^-) = f'(0^+)$

∴ f(x) is differentiable at x = 0.

(iii) $f'(1^-) = \lim\limits_{h \to 0} \frac{f(1-h)-f(1)}{-h}$

$= \lim\limits_{h \to 0} \frac{\sin\left\{\frac{\pi(1-h)}{-h}\right\} - 1}{-h} = \lim\limits_{h \to 0} \frac{\cos\frac{\pi h}{2} - 1}{-h}$

$= \lim\limits_{h \to 0} \frac{2\sin^2\frac{\pi h}{4}}{\left(\frac{\pi h}{4}\right)^2} \cdot \frac{\pi^2 h}{16} = \lim\limits_{h \to 0} \left(\frac{\pi^2 h}{8}\right) = 0$

$f'(1^+) = \lim\limits_{h \to 0} \frac{f(1+h)-f(1)}{h} = \lim\limits_{h \to 0} \frac{|-1+h|-1}{h}$

$= \lim\limits_{h \to 0} \frac{|-1+h|-1}{h} = \lim\limits_{h \to 0} \frac{1-h-1}{h} = -1$

$f'(1^-) \ne f'(1^+)$

Hence f(x) is not differentiable at x = 1 but it is continuous there.

(iv) $f'(3/2^-) = \lim\limits_{h \to 0} \frac{f\left(\frac{3}{2}-h\right) - f\left(\frac{3}{2}\right)}{-h}$

$= \lim\limits_{h \to 0} \frac{|3-2h-3|\left[\frac{3}{2}-h\right] - 0}{-h}$

$= \lim\limits_{h \to 0} \frac{2h\,.\,1}{-h} = -2$

$f'(3/2^+) = \lim\limits_{h \to 0} \frac{f\left(\frac{3}{2}+h\right) - f\left(\frac{3}{2}\right)}{h}$

$= \lim\limits_{h \to 0} \frac{|2h|\left[\frac{3}{2}+h\right] - 0}{h} = \lim\limits_{h \to 0} \frac{2h\,.\,1}{h} = 2$

∴ f(x) is not differentiable at x = $\frac{3}{2}$ but it is continuous there.

Problem 5.

Let $f(x)=\begin{cases} \left(\dfrac{(1+\{x\})^{\frac{1}{\{x\}}}}{e}\right)^{\frac{1}{\{x\}}}, & x \ne \text{Integer} \\ \dfrac{2}{e}, & x = \text{Integer} \end{cases}$.

Discuss the continuity and differentiability of f(x) at any integral point, where {.} denotes the fractional part.

Solution Let $x = I_0$ be any arbitrary integers.

$f(I_0) = \frac{2}{e}$.

$f(I_0^{\,-}) = \lim\limits_{h\to0} f(I_0-h)$

$= \lim\limits_{h\to0}\left(\frac{(1+\{I_0-h\})^{\frac{1}{\{I_0-h\}}}}{e}\right) = \left(\frac{(1+1)^1}{e}\right)^1 = \frac{2}{e}$

$f(I_0+h) = \lim\limits_{h\to0}(I_0+h) = \lim\limits_{h\to0}\left(\frac{(1+h)^{1/h}}{e}\right)^{1/h}$

$= e^{\lim\limits_{h\to0}\left(\frac{(1+h)^{1/h}}{e}\right)^{1/h}} = e^{\lim\limits_{h\to0}\frac{1}{h}\left(\frac{(1+h)^{1/h}}{e}-1\right)}$

$= e^{\lim\limits_{h\to0}\frac{1}{h}\left(\frac{(1+h)^{1/h}}{e}-1\right)} = e^{\lim\limits_{h\to0}\frac{(1+h)^{1/h}-e}{eh}}$

Now $(1+h)^{1/h}$

$$= e^{\frac{1}{h}\ln(1+h)} = e^{\frac{1}{h}\left(h-\frac{h^2}{2}+\frac{h^3}{3}-....\right)} = e.e^{-h\left(\frac{1}{2}-\frac{h}{3}+\frac{h^2}{4}-.....\right)}$$

$$\Rightarrow \quad f(I_0{}^{+}) = e^{\lim\limits_{h\to 0}\frac{e^{-h\left(\frac{1}{2}-\frac{h}{3}+\frac{h^2}{4}-....\right)}-1}{h}}$$

$$= e^{\lim\limits_{h\to 0}\frac{\left(e^{-h\left(\frac{1}{2}-\frac{h}{3}+\frac{h^2}{4}-....\right)}-1\right)\left(\frac{1}{2}-\frac{h}{3}+\frac{h^2}{4}-....\right)(-1)}{-h\left(\frac{1}{2}-\frac{h}{3}+\frac{h^2}{4}-....\right)}} = e^{-\frac{1}{2}}$$

Since $f(I_0{}^{+}) \neq f(I_0{}^{-})$

$\Rightarrow$ f(x) is discontinuous at all integral points and hence non-differentiable.

Problem 6. Let

$$f(x) = \begin{cases} \left(\ell n\left(e\left|[x]+[-x]\right|\right)^x\right)\cdot\left(\dfrac{2e^{\left(\frac{\{x\}+\{-x\}}{|x|}\right)}-5}{3+e^{\frac{1}{|x|}}}\right) & \text{for } x<0 \\ 0 & \text{for } x=0 \\ x\cdot\dfrac{1-e^{|x|+\{x\}}}{|x|+\{x\}} & \text{for } x>0 \end{cases}$$

where [ ], { } represents integral and fractional part functions respectively. Compute the one-sided derivatives at $x = 0$ and comment on the continuity and differentiability at $x = 0$.

Solution $$f'(0^+) = \lim_{h\to 0}\frac{h\left(\frac{1-e^{h+h}}{h+h}\right)-0}{h}$$

$$= \lim_{h\to 0}\frac{(1-e^{2h})}{(2h)h}h = -1$$

$$f'(0^-) = \lim_{h\to 0}\frac{-h\cdot \ell n\left(e\left|-1+0\right|\right)\cdot\left(\dfrac{2e^{\frac{1-h+h}{|-h|}}-5}{3+e^{-\frac{1}{h}}}\right)}{-h}$$

$$= \lim_{h\to 0}\frac{-h\cdot\left(\dfrac{2e^{\frac{1}{h}}-5}{3+e^{\frac{1}{h}}}\right)}{-h}$$

$$= \lim_{h\to 0}\frac{2-5e^{-\frac{1}{h}}}{3e^{-\frac{1}{h}}+1} = 2$$

Hence f is continuous but not derivable at $x = 0$.

Problem 7. Let f be a continuous function from R to R such that f is differentiable at 0. Suppose that $f(1/n) = 0$ for all $n \in N$.

(i) Prove that $f(0) = 0$. (This fact does not requires the differentiability of f)

(ii) Prove that $f'(0) = 0$.

Solution

(i) Since f is continuous at 0 and $\lim\limits_{n\to\infty}\left(\frac{1}{n}\right) = 0$,

we must have $f(0) = \lim\limits_{n\to\infty} f\left(\frac{1}{n}\right) = 0$,

since we are given that $f\left(\frac{1}{n}\right) = 0$ for all $n \in N$.

(ii) By part (i) we have $f(0) = 0$ and so

$$f'(0) = \lim_{x\to 0}\left(\frac{f(x)-f(0)}{x-0}\right) = \lim_{x\to 0}\left(\frac{f(x)}{x}\right)$$

Thus, since $\frac{1}{n} \neq 0$ and $\frac{1}{n} \to 0$ as $n \to \infty$,

$$f'(0) = \lim_{n\to\infty}\left(\frac{f\left(\frac{1}{n}\right)}{\frac{1}{n}}\right) = 0,$$

because $f\left(\frac{1}{n}\right) = 0$ for all $n \in N$.

Problem 8. If $f(x) = 3x^{10} - 7x^8 + 5x^6 - 21x^3 + 3x^2 - 7$ then find the value of $\lim\limits_{x\to 1^+}\dfrac{f(1-h)-f(1)}{h^3+3h}$.

Solution $f(x) = 3x^{10} - 7x^8 + 5x^6 - 21x^3 + 3x^2 - 7$

$$l = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}\cdot\frac{-h}{h(h^2+3)}$$

$$l = \lim_{h\to 0} -f'(1)\cdot\frac{1}{h^2+3} = -\frac{1}{3}f'(1) \quad ...(1)$$

Now, $f'(x) = 30x^9 - 56x^7 + 30x^5 - 63x^2 + 6x$

$$f'(1) = 30 - 56 + 30 - 63 + 6 = -53$$

$\therefore$ From (1), $l = -\frac{1}{3}(-53) = \frac{53}{3}$.

Problem 9. If $f(x+y)=f(x)+f(y)+|x|y+xy^2$, $\forall\ x, y \in R$ and $f'(0)=0$, then prove that f is differentiable for all $x \in R$, but it is not twice differentiable at $x=0$.

Solution

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f(h)+|x|h+xh^2}{h}$$

$\because$ $f(0)=0$, we have

$$f'(x)=\lim_{h\to 0}\left(\frac{f(h)-f(0)}{h}+|x|+xh\right)$$

$$f'(x)=f'(0)+|x|=|x|.$$

Hence, f is differentiable for all $x \in R$.

It is easily seen that $f'(x) = |x|$ is not differentiable at $x=0$. So, f is not twice differentiable at $x=0$.

Problem 10. If the function $f(x)=|x-a|.g(x)$, where g(x) is a continuous function, is differentiable at x = a then find g(a).

Solution We have

$$f'(a^+)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{hg(a+h)-0}{h}=g(a).$$

[$\because$ g is continuous at x = a we have $\lim_{h\to 0} g(a+h)=\lim_{h\to 0} g(a-h)=g(a)$]

$$f'(a^-)=\lim_{h\to 0}\frac{f(a)-f(a-h)}{h}=\lim_{h\to 0}\frac{0-hg(a-h)}{h}=-g(a)$$

Since, f is differentiable at x = a, therefore we have

$g(a)=-g(a)$

$\Rightarrow$ $g(a)=0.$

Problem 11. If $f(x)=\begin{cases}\frac{2}{3}(a^2-b^2), & 0\le x\le b\\ \frac{2}{3}a^2-\frac{4}{9}x^2-\frac{2b^3}{9x}, & b<x\le a,\\ \frac{2}{9x}(a^3-b^3), & x>a\end{cases}$

then find whether $f'(a)$ exists.

Solution We have,

$$f'(x)=\begin{cases}0, & 0\le x<b\\ \frac{-8}{9}x+\frac{2b^3}{9x^2}, & b<x<a,\\ \frac{-2}{9x^2}(a^3-b^3), & x>a\end{cases}$$

$$\therefore\quad f'(a^+)=\lim_{h\to 0}f'(a+h)=\lim_{h\to 0}\left[\frac{-2}{9}\cdot\frac{(a^3-b^3)}{(a+h)^2}\right]$$

$$=\frac{-2a^3+2b^3}{9a^2}.$$

and $f'(a^-)=\lim_{h\to 0}f'(a-h)$

$$=\lim_{h\to 0}\left[\frac{-8}{9}(a-h)+\frac{2b^3}{9(a-h)^2}\right]$$

$$=\frac{-8a}{9}+\frac{2b^3}{9a^2}=\frac{-8a^3+2b^3}{9a^2}.$$

$\therefore$ $f'(a^+)\ne f'(a^-)$ and hence $f'(a)$ does not exist.

Problem 12. Let $f:[0,1]\to R$ be defined as

$$f(x)=\begin{cases}x^3(1-x)\sin\left(\frac{1}{x^2}\right) & \text{if } 0<x\le 1\\ 0 & \text{if } x=0\end{cases}.$$

Then prove that (i) f(x) is differentiable in [0, 1]

(i) f(x) is bounded in [0, 1]

(ii) f′(x) is bounded in [0, 1].

Solution $$f'(0^+)=\lim_{h\to 0}\frac{h^3(1-h)\sin\frac{1}{h^2}-0}{h}=0$$

$$f'(1^-)=\lim_{h\to 0}\frac{(1-h)^3(+h)\sin\frac{1}{(1-h)^2}-0}{-h}$$

$$=\lim_{h\to 0}-(1-h)^3\sin\frac{1}{(1-h)^2}=-\sin 1$$

Hence f is derivable in [0, 1].

Obviously f is continuous in a closed interval [0, 1] and hence f is bounded.

Now f′(x)

$$=\begin{cases}(x^3-x^4)\cos\left(\frac{1}{x^2}\right)\left(-\frac{2}{x^3}\right)+\sin\frac{1}{x^2}(3x^2-4x^3) & x\ne 0\\ 0 \quad \text{if } x=0\end{cases}$$

and $\lim_{x\to 1^-}f'(x)=(0)+\sin 1(3-4)=-\sin 1$.

Thus, f′(x) is continuous in [0, 1] and hence it is also bounded.

Problem 13. $f(x)=\begin{cases}x+a & \text{if } x<0\\ |x-1| & \text{if } x\ge 0\end{cases}$ and

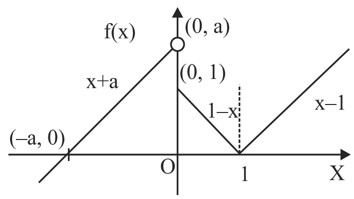
$$g(x)=\begin{cases}x+1 & \text{if } x<0\\ (x-1)^2+b & \text{if } x\ge 0\end{cases}$$

where a and b are non negative real numbers.

Determine the composite function gof. If (gof) (x) is continuous for all real x, determine the values of a and b. Further, for these values of a and b, is gof differentiable at x = 0 ? Justify your answer.

Solution $f(x)=\begin{cases} x+a & \text{if } x<0 \\ |x-1| & \text{if } x\ge 0\end{cases}$ and

$$g(x)=\begin{cases} x+1 & \text{if } x<0 \\ (x-1)^2+b & \text{if } x\ge 0\end{cases}$$



$$G(x)=g(f(x))=\begin{cases} f(x)+1 & f(x)<0 \\ (f(x)-1)^2+b & f(x)\ge 0\end{cases}$$

$$=\begin{cases} x+a+1 & \text{if } f(x)<0 \text{ or } x<-a \\ (x+a-1)^2+b & \text{if } f(x)\ge 0 \text{ or } -a<x<0 \\ (1-x-1)^2+b = x^2+b & \text{if } f(x)\ge 0 \text{ or } 0\le x<1 \\ (x-1-1)^2+b = (x-2)^2+b & \text{if } f(x)\ge 0 \text{ or } x\ge 1\end{cases}$$

(gof) (x) is continuous :

Continuity at $x=-a$

$$\lim_{h\to 0} G(-a-h) = \lim_{h\to 0}(-a-h)+a+1 = 1$$

$$\lim_{h\to 0} G(-a+h) = \lim_{h\to 0}(-a+h+a-1)^2+b = b+1$$

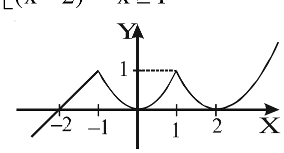
$\Rightarrow$ $b+1=1 \Rightarrow b=0.$

Continuity at $x=0$

$$\lim_{h\to 0} G(0-h) = (a-1)^2 \qquad (\text{as } b=0)$$

$$\lim_{h\to 0} G(0+h) = 0 \Rightarrow a-1=0 \Rightarrow a=1$$

$$\therefore\quad G(x)=\begin{cases} x+2 & x<-1 \\ x^2 & -1\le x<1 \\ (x-2)^2 & x\ge 1\end{cases}$$



$\Rightarrow$ G is differentiable at x = 0.

Problem 14. Let $f : R \to R$ be a real valued function $\forall\, x, y \in R$ such that $|f(x)-f(y)| < \sin^4(x-y)$. Prove that f(x) is a constant function.

Solution Since $|f(x)-f(y)| < \sin^4(x-y)$,

$x \ne y\, |f(x)-f(y)| < |\sin^4(x-y)|$

Dividing by | x – y | on both sides,

$$\therefore\quad \left|\frac{f(x)-f(y)}{x-y}\right| < \left|\frac{\sin^4(x-y)}{x-y}\right|$$

Taking lim $y \to x$, we get using domination law of limits that

$$\lim_{y\to x}\left|\frac{f(x)-f(y)}{x-y}\right| \le \lim_{y\to x}\left|\frac{\sin^4(x-y)}{x-y}\right|$$

$$\Rightarrow\quad \left|\lim_{y\to x}\frac{f(x)-f(y)}{x-y}\right| \le \left|\lim_{y\to x}\frac{\sin^4(x-y)}{x-y}\right|$$

$\Rightarrow$ $|f'(x)| \le 0$

$\Rightarrow$ $|f'(x)| = 0$ $\quad (\because |f'(x)| \ge 0, \text{ in general})$

$\therefore$ $f'(x)=0$

$\Rightarrow$ $f(x) = c$ (constant)

Problem 15. A function $f : R \to [1, \infty)$ satisfies the equation $f(xy) = f(x)f(y) - f(x) - f(y) + 2$. If f is differentiable on R and f(2) = 5, then show that

$$f'(x)=\frac{f(x)-1}{x}.f'(1) \text{ .Hence, determine } f(x).$$

Solution We have

$$f(xy)=f(x)f(y)-f(x)-f(y)+2 \qquad ...(1)$$

$$\text{Now, } f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f\left\{x\left(1+\frac{h}{x}\right)\right\}-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x)f\left(1+\frac{h}{x}\right)-f(x)-f\left(1+\frac{h}{x}\right)+2-f(x)}{h}$$

[using (1)]

$$=\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)-2}{\frac{h}{x}}.\frac{f(x)-1}{x} \qquad ...(2)$$

Putting x = 1 and y = 2 in equation (1), we have

$f(2) = f(1)\, f(2) - f(1) - f(2) + 2$

$\Rightarrow 5 = 5(f(1) - f(1) - 5 + 2$ $\qquad [\because f(2) = 5]$

$\Rightarrow f(1) = 2.$

Now (2) reduces to

$$f'(x)=\frac{f(x)-1}{x}.\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)-f(1)}{\frac{h}{x}}=\frac{f(x)-1}{x}.f'(1)$$

To find f(x), write the above equation as

$$\frac{df}{f-1}=f'(1)\frac{dx}{x}$$

Integrating both sides w.r.t. x

we get $\ln|f-1|=f'(1).\ln|x|+\ln c$.

Let $f'(1)=a$

$\ln|f-1|=a.\ln|x|c$

Now, using the condition $f(1)=2$, we have

$\ln|2-1|=a.\ln c \Rightarrow c=1$

Now, $f-1=\pm|x|^a$

$\Rightarrow f(x)=1\pm|x|^a$

Using the condition $f(2)=5$, we have

$5=1\pm 2^a$

This gives $a=2$, using + sign.

Hence, we have $f(x)=1+|x|^2$

which can also be written as $f(x)=1+x^2$.

Problem 16. Let $f\left(\frac{x_1+x_2+x_3+....+x_n}{n}\right)$

$$=\left(\frac{f(x_1)+f(x_2)+f(x_3)+....+f(x_n)}{n}\right)$$

where x_i are any real numbers and $n\in N$. If f(x) is differentiable and $f'(0)=a$ and $f(0)=b$, find f(x).

Solution We have, $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$

$\{$taking $x_1=x, x_2=y, n=2\}$

This holds for any real x, y. So y is independent of x, i.e., $\frac{dy}{dx}=0$.

Differentiating w.r.t x,

$$f'\left(\frac{x+y}{2}\right)\cdot\frac{1}{2}\left(1+\frac{dy}{dx}\right)=\frac{1}{2}\left\{f'(x)+f'(y)\frac{dy}{dx}\right\}$$

$$\therefore\quad \frac{1}{2}f'\left(\frac{x+y}{2}\right)=\frac{1}{2}f'(x)$$

Taking x = 0 and x in place of y we get

$$\frac{1}{2}f'\left(\frac{0+x}{2}\right)=\frac{1}{2}f'(0)$$

$$\therefore\quad f'\left(\frac{x}{2}\right)=f'(0)=a\,(\text{given}).$$

Integrating w.r.t. $\frac{x}{2}$, $f\left(\frac{x}{2}\right)=a.\frac{x}{2}+c$

When $x=0$, $f(0)=0+c$, i.e., $b=c$

$[\because f(0)=b\ (\text{given})]$

Thus, $f\left(\frac{x}{2}\right)=\frac{ax}{2}+b$ ...(1)

Again, putting $y=0$ in $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$ we get

$$f\left(\frac{x}{2}\right)=\frac{f(x)+f(0)}{2}=\frac{f(x)+b}{2}$$

$\therefore$ From (1), $\frac{f(x)+b}{2}=\frac{ax}{2}+b$

$\therefore$ $f(x)=ax+b$.

Problem 17. Find a function continuous and derivable for all x and satisfying the functional relation, $f(x+y).f(x-y)=f^2(x)$, where x and y are independent variables and $f(0)\neq 0$.

Solution Put $y=x$ and $x=0$ to get

$f(x).f(-x)=f^2(0)$ ...(1)

$$\text{Now}\quad f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)-\frac{f^2(0)}{f(-x)}}{h}$$

$$=\frac{1}{f(-x)}.\lim_{h\to 0}\frac{f(x+h).f(-x)-f^2(0)}{h}$$

$$=\frac{1}{f(-x)}\lim_{h\to 0}\frac{f\left(\frac{h}{2}+\left(\frac{h}{2}+x\right)\right).f\left(\frac{h}{2}-\left(\frac{h}{2}+x\right)\right)-f^2(0)}{h}$$

$$=\frac{1}{f(-x)}\lim_{h\to 0}\frac{f^2\left(\frac{h}{2}\right)-f^2(0)}{h}$$

$$=\frac{1}{2f(-x)}\lim_{h\to 0}\frac{\left[f\left(\frac{h}{2}\right)+f(0)\right]\left[f\left(\frac{h}{2}\right)-f(0)\right]}{\frac{h}{2}}$$

$$=\frac{f(x)}{2f^2(x)}.2f(0).f'(0)=\frac{f'(0)}{f(0)}f(x)$$

$$\Rightarrow\quad \frac{f'(x)}{f(x)}=\frac{f'(0)}{f(0)}=k\ (\text{say})$$

$\Rightarrow$ $\ln|f(x)|=kx+c$

$\Rightarrow$ $|f(x)|=e^{kx+c}=e^{kx}\cdot e^c$

$\Rightarrow$ $f(x)=\pm e^{kx}\cdot e^c$

$\Rightarrow$ $f(x)=a.e^{kx}$ where a and k are arbitrary constants.

Problem 18. Let f be a function such that

$f(x+f(y)) = f(f(x)) + f(y) \quad \forall\ x, y \in R$

and $f(h) = h$ for $0 < h < \varepsilon$ where ε is a small positive quantity. Determine f''(x) and f(x).

Solution Given $f(x+f(y)) = f(f(x)+f(y))$...(1)

Putting $x = y = 0$ in (1),

$f(0+f(0)) = f(f(0)) + f(0) \Rightarrow f(f(0)) = f(f(0)) + f(0)$

$\therefore\ f(0) = 0$ ...(2)

Now $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ (for $0 < h < \varepsilon$)

putting x as h and y as x in (1)

$= \lim_{h\to 0}\frac{f(f(h))}{h} = \lim_{h\to 0}\frac{f(h)}{h}$ $(\because f(h) = h)$

$= \lim_{h\to 0}\frac{h}{h} = 1.$

Integrating both sides with limits 0 to x

$f(x) = x$

$\therefore\ f''(x) = 0.$

Problem 19. Let f(x) be a thrice differentiable function satisfying $f(x+y) = f(x-y) + y[f'(x+y) + f'(x-y)]$ where $f(0) = 0$, $f'(0) = 1$, and $f(1) = 2$. Find f(x).

Solution Put $y = x$

$f(2x) = f(0) + x[f'(2x) + f'(0)]$

$f(2x) = xf'(2x) + xf'(0) + f(0)$

Differentiate both sides w.r.t. x

$f'(2x).\,2 = f'(2x) + xf''(2x).2 + f'(0)$

$f'(2x) = 2xf''(2x) + f'(0)$

Replace 2x by x,

$f'(x) = xf''(x) + f'(0)$

Again differentiate both sides w.r.t. x

$f''(x) = xf'''(x) + f''(x)$

$\therefore\ xf''(x) = 0$

$\Rightarrow\ f'''(x) = 0$

$\therefore\ f(x) = ax^2 + bx + c$

Since $f(0) = 0$, $c = 0$

$f'(0) = 1 \Rightarrow b = 1$ and $f(1) = 2 \Rightarrow a = 1.$

Hence $f(x) = x^2 + x.$

Problem 20. Let f(x) be a positive differentiable function satisfying

$f\left(\frac{x+y}{2}\right) = \sqrt{f(x).f(y)}\ \forall\ x, y \in R$, where $f'(0) = 2.$

Find f(x).

Solution Put $x = y = 0$, $f(0) = \sqrt{f(0)^2}$

$\Rightarrow\ f(0) = 0, 1$

Since f is a positive function $f(0) = 1.$

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$= \frac{\sqrt{f(2x).f(2h)} - f(x)}{h}$

Replace x by 2x and y = 0 in the rule :

$f(x) = \sqrt{f(2x).f(0)}$

$f'(x) = \lim_{h\to 0}\frac{f(x)\sqrt{\frac{f(2h)}{f(0)}} - f(x)}{h}$

$= \lim_{h\to 0}\frac{f(x)(\sqrt{f(2h)}-1)}{h}$

$= f(x).\lim_{h\to 0}\frac{(f(2h)-1)}{h.(\sqrt{f(2h)}+1)}$

$= f(x).\lim_{h\to 0}\frac{f(2h)-1}{2h}$

$f'(x) = f(x).f'(0)$

$f'(x) = 2f(x)$

$\ell n\, f(x) = 2x + c$

$f(x) = e^{2x+c}$

$f(0) = 1 \Rightarrow c = 0$

$\therefore\ f(x) = e^{2x}$

Things to Remember

1. Right Hand Derivative (R.H.D.)

$= f'(a^+) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

Left Hand Derivative (L.H.D.)

$= f'(a^-) = \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$

f is differentiable at $x = a$ if and only if L.H.D. = R.H.D.

2. A function will fail to have a derivative at a point where the graph has

(i) a corner, where the one-sided derivatives differ.

(ii) an oscillation point, where the one-sided derivative(s) does (do) not exist.
(iii) a vertical tangent, where the absolute value of slope of the secant approaches ∞.
(iv) a discontinuity.

3. The curve $y = f(x)$ has a vertical tangent line at the point $(x_0, f(x_0))$ provided that f is continuous at x_0 and $|f'(x)| \to \infty$ as $x \to x_0$

4. The graph of a continuous function f has a cusp at x_0 if $f'(x)$ approaches ∞ from one side and $-\infty$ from the other side.

5. If a function f is differentiable at $x = a$ then it must be continuous at $x = a$.
If f is continuous at $x = a$, then f may or may not be differentiable at $x = a$.

6. If a function f is discontinuous at $x = a$ then it is non-differentiable at $x = a$.

7. If a function f is non-differentiable at $x = a$ but both the one-sided derivatives exist (though being unequal), then f is continuous at $x = a$.

8. If a function f is defined only in the left neighbourhood of a point $x = a$ (for $x \le a$), it is said to be differentiable at a if

L.H.D. $= f'(a^-) = \lim_{h\to 0} \dfrac{f(a-h)-f(a)}{-h}$ exists. In such a case $f'(a) = f'(a^-)$.

Similarly, if a function f is defined only in the right neighbourhood of a point $x = a$ (for $x \ge a$), it is said to be differentiable at a if

R.H.D. $= f'(a^+) = \lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ exists.

In such a case $f'(a) = f'(a^+)$.

9. A function f(x) is said to be differentiable in an open interval (a, b) if it is differentiable at all interior points in (a, b).

10. A function f(x) is said to be differentiable in a closed interval [a, b] if
(i) it is differentiable at all interior points in (a, b).
(ii) R.H.D. $= f'(a^+)$ exists at the left endpoint a.
(ii) L.H.D. $= f'(a^-)$ exists at the right endpoint b.

11. (i) All polynomial, exponential, logarithmic and trigonometric functions (inverse trigonometric not included) are differentiable at each point in their domain.
(ii) Modulus function and signum function are non differentiable at $x = 0$. Hence, $y = |f(x)|$ and $y = \text{sgn}(f(x))$ should be checked at points where $f(x) = 0$.
(iii) Power function $y = x^p$, $0 < p < 1$ is non-differentiable at $x = 0$. Hence, $y = (f(x))^p$ should be checked at points where $f(x) = 0$.
(iv) The inverse trigonometric functions $y = \sin^{-1}x$, $\cos^{-1}x$, $\text{cosec}^{-1}x$, and $\sec^{-1}x$ are not differentiable at the points $x = \pm 1$. Hence, $y = \sin^{-1}(f(x))$, $\cos^{-1}(f(x))$, $\text{cosec}^{-1}(f(x))$, and $\sec^{-1}(f(x))$ should be checked at points where $f(x) = \pm 1$.
(v) Greatest integer function and fractional part functions are non differentiable at all integral x. Hence, $y = [f(x)]$ and $y = \{f(x)\}$ should be checked at points where $f(x) = n$, $n \in I$.
(vi) Further, a function should be checked at all those points where discontinuity may arise.

12. An alternative limit form of the derivative
$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}.$$

13. If $f'(a)$ exists, then $\lim_{h\to 0} \dfrac{f(a+h)-f(a-h)}{2h} = f'(a)$.

14. (i) If $f'(a)$ exists and $\psi(h) \to 0$ as $h \to 0$, then
$$\lim_{h\to 0} \frac{f(a+\psi(h))-f(a)}{\psi(h)} = f'(a)$$
(ii) If $f'(a)$ exists, and $\psi(h) \to 0$ and $\phi(h) \to 0$ as $h \to 0$, then $\lim_{h\to 0} \dfrac{f(a+\psi(h))-f(a+\phi(h))}{\psi(h)-\phi(h)}$
$= f'(a)$

15. Let the function $y = f(x)$ be defined by $x = x(t)$ and $y = y(t)$, where t is the parameter.
Then, $$\frac{dy}{dx} = \lim_{\delta\to 0} \frac{\frac{y(t+\delta)-y(t)}{\delta}}{\frac{x(t+\delta)-x(t)}{\delta}} = \lim_{\delta\to 0} \frac{y(t+\delta)-y(t)}{x(t+\delta)-x(t)}$$

16. In general, the limit of the derivative may not exist, but when it exists then it is equal to the value of the derivative.
In such a case, we say that the derived function $f'(x)$ is continuous, or the function f(x) is continuously differentiable.

17. (i) If $\lim_{x\to a^-} f'(x)$ and $\lim_{x\to a^+} f'(x)$ both exist or are infinite then $\lim_{x\to a^-} f'(x) = f'(a^-)$

i.e L.H.L. of $f'(x)$ = L.H.D. of f(x) at x = a

and $\lim_{x\to a^+} f'(x) = f'(a^+)$

i.e R.H.L. of $f'(x)$ = R.H.D. of f(x) at x = a.

(a) If $\lim_{x\to a^-} f'(x) = \lim_{x\to a^+} f'(x)$, then f(x) is differentiable and its derivative $f'(x)$ is continuous at x = a.

(b) If $\lim_{x\to a^-} f'(x) \neq \lim_{x\to a^+} f'(x)$, then f(x) is non-differentiable and its derivative $f'(x)$ is discontinuous at x = a.

(c) If $\lim_{x\to a^-} f'(x)$ and $\lim_{x\to a^+} f'(x)$ are infinite, then f(x) has an infinite derivative at x = a and hence it is non-differentiable there and $f'(x)$ is discontinuous at x = a.

(ii) If any of the limits $\lim_{x\to a^-} f'(x)$ or $\lim_{x\to a^+} f'(x)$ does not exist, then we cannot conclude anything about the differentiability of the function. In such a case, we should try to find the derivative using its basic definition (i.e. first principles).

18. A function f(x) is twice differentiable at x = a if its derivative $f'(x)$ is differentiable at x = a i.e.

the limit $f''(a) = \lim_{h\to 0}\frac{f'(a+h)-f'(a)}{h}$ exists.

19. If f(x) and g(x) are differentiable at x = a, then the following functions are also differentiable at x = a.

(i) cf(x) is differentiable at x = a, where c is any constant.

(ii) f(x) ± g(x) is differentiable at x = a.

(iii) f(x). g(x) is differentiable at x = a.

(iv) f(x)/g(x) is differentiable at x = a, provided $g(a) \neq 0$.

20. If f(x) is differentiable at x = a and g(x) is non-differentiable at x = a, then we have the following results :

(i) Both the functions f(x) + g(x) and f(x) – g(x) are non-differentiable at x = a.

(ii) f(x). g(x) is not necessarily non-differentiable at x = a. We need to find the result by first principles.

21. If f(x) and g(x) both are non-differentiable at x = a, then we have the following results.

(i) The functions f(x) + g(x) and f(x) – g(x) are not necessarily non-differentiable at x = a. However, atmost one of f(x) + g(x) or f(x) – g(x) can be differentiable at x = a. That is, both of them cannot be differentiable simultaneously at x = a.

(ii) f(x). g(x) and f(x)/g(x) are not necessarily non-differentiable at x = a. We need to find the result by applying first principles.

22. If f(x) is differentiable at x = a and g(x) is differentiable at x = f(a) then the composite function (gof)(x) is differentiable at x = a.

Objective Exercises

SINGLE CORRECT ANSWER TYPE

1. If $f(x) = \frac{x}{1+e^{1/x}}$, $x \neq 0$ and f(0) = 0 then,

(A) f(x) is continuous at x = 0 and $f'(x) = 1$

(B) f(x) is discontinuous at x = 0

(C) f(x) is continuous at x = 0 and $f'(x)$ does not exists

(D) f(x) is continuous at x = 0 and $f'(x) = 0$

2. Let $f(x) = \begin{cases} x^2 & \text{if } x \le x_0 \\ ax+b & \text{if } x > x_0 \end{cases}$

The values of the coefficients a and b for which the function is continuous and has a derivative at x_0, are

(A) $a = x_0, b = -x_0$ (B) $a = 2x_0, b = -x_0^2$

(C) $a = a = x_0^2, b = -x_0$ (D) $a = x_0, b = -x_0^2$

3. The number of points where function $f(x) = \text{minimum } \{x^3 - 1, -x + 1, \text{sgn }(-x)\}$ is continuous but not differentiable is

(A) One (B) Two

(C) Zero (D) None of these

4. Total number of the points where the function $f(x) = \min\{|x| - 1, |x - 2| - 1|$ is not differentiable

(A) 3 points (B) 4 points

(C) 5 points (D) None of these

5. The set of values of x for which the function defined as

$$f(x)=\begin{cases}1-x & x<1\\ (1-x)(2-x) & 1\le x\le 2\\ 3-x & x>2\end{cases}$$

fails to be continuous or differentiable, is

(A) $\{1\}$ (B) $\{2\}$
(C) $\{1,2\}$ (D) ϕ

6. The number of points where $f(x)=(x+1)^{2/3}+|x-1|^{\sqrt{3}}$, is non-differentiable is

(A) 1 (B) 2
(C) 3 (D) none

7. The set of all points where $f(x)=\sqrt[3]{x^2|x|}-|x|-1$ is not differentiable is

(A) $\{0\}$ (B) $\{-1,0,1\}$
(C) $\{0,1\}$ (D) none of these

8. It f(x) is a differentiable function from $R\rightarrow Q$ then $\sum_{r=0}^{100}(-1)^r f(r)$ is

(A) 0 (B) –1
(C) 1 (D) none

9. The function $f(x)=\dfrac{|x|-x(3^{1/x}+1)}{3^{1/x}-1}$, $x\neq 0$, $f(0)=0$ is

(A) discontinuous at x = 0
(B) continuous at x = 0 but not differentiable there
(C) both continuous and differentiable at x = 0
(D) differentiable but not continuous at x = 0

10. Let $f(x)=\begin{cases}g(x).\cos\dfrac{1}{x} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$ where g(x) is an even function differentiable at x = 0, passing through the origin. Then $f'(0)$

(A) is equal to 1 (B) is equal to 0
(C) is equal to 2 (D) does not exist

11. Let $f(x)=\lim_{n\to\infty}\dfrac{(x^2+2x+3+\sin\pi x)^n-1}{(x^2+2x+3+\sin\pi x)^n+1}$, then

(A) f(x) is continuous and differentiable for all $x\in R$.
(B) f(x) is continuous but not differentiable for all $x\in R$.
(C) f(x) is discontinuous at infinite number of points.
(D) f(x) is discontinuous at finite number of points.

12. Let f (x) is a function continuous for all $x\in R$ except at x = 0 such that $f'(x)<0\ \forall\ x\in(-\infty,0)$ and $f(x)>0\ \forall\ x\in(0,\infty)$. If $\lim_{x\to 0^+}f(x)=3$, $\lim_{x\to 0^-}f(x)=4$ and $f(0)=5$, then the image of the point (0,1) about the line $y\lim_{x\to 0}f(\cos^3 x-\cos^2 x)=x\lim_{x\to 0}f(\sin^2 x-\sin^3 x)$, is

(A) $\left(\dfrac{12}{25},\dfrac{-9}{25}\right)$ (B) $\left(\dfrac{12}{25},\dfrac{9}{25}\right)$
(C) $\left(\dfrac{16}{25},\dfrac{-8}{25}\right)$ (D) $\left(\dfrac{24}{25},\dfrac{-7}{25}\right)$

13. Let f be an injective and differentiable function such that $f(x).f(y)+2=f(x)+f(y)+f(xy)$ for all non negative real x and y with $f'(0)=0$, $f'(1)=2\neq f(0)$, then

(A) $xf'(x)-2f(x)+2=0$
(B) $xf'(x)+2f(x)-2=0$
(C) $xf'(x)-f(x)+1=0$
(D) $2f(x)=f'(x)+2$

14. Let f be a function such that $f(x+y)=f(x)+f(y)$ for all x and y and $f(x)=(2x^2+3x)\,g(x)$ for all x where g(x) is continuous and $g(0)=3$. Then $f'(x)$ is equal to

(A) 9 (B) 3
(C) 6 (D) none

15. Let f(x) be a function such that $f(x+y)=f(x)+f(y)$ and $f(x)=\sin x\, g(x)$ for all $x,y\in R$. If g(x) is a continuous function such that $g(0)=K$, then $f'(x)$ is equal to

(A) K (B) Kx
(C) Kg(x) (D) none

16. Let $f(x)=\begin{cases}\dfrac{3x^2+2x-1}{6x^2-5x+1} & \text{for } x\neq\dfrac{1}{3}\\ -4 & \text{for } x=\dfrac{1}{3}\end{cases}$ then $f'\left(\dfrac{1}{3}\right)$

(A) is equal to –9 (B) is equal to –27
(C) is equal to 27 (D) does not exist

17. Given f(x) is a differentiable function of x, satisfying $f(x).f(y)=f(x)+f(y)+f(xy)-2$ and that $f(2)=5$. Then f(3) is equal to

(A) 10 (B) 24
(C) 15 (D) none

18. The function $f(x)=\begin{bmatrix} 2x+1 & ,x\in Q \\ x^2-2x+5 & ,x\notin Q \end{bmatrix}$ is

(A) continuous no where
(B) differentiable no where
(C) continuous but not differentiable exactly at one point
(D) differentiable and continuous only at one point and discontinuous elsewhere

19. Let f(x) be differentiable at x = h then $\lim_{x\to h}\frac{(x+h)\,f(x)-2h\;f(h)}{x-h}$ is equal to

(A) $f(h)+2hf'(h)$ (B) $2\,f(h)+hf'(h)$
(C) $hf(h)+2f'(h)$ (D) $hf(h)-2f'(h)$

20. If $f(x).f(y)=f(x)+f(y)+f(xy)-2\ \forall\ x,y\in R$ and if f(x) is not a constant function, then the value of f(1) is

(A) 1 (B) 2
(C) 0 (D) – 1

21. If $f(x)=|1-x|$, then the points where $\sin^{-1}(f|x|)$ is non-differentiable are

(A) {0, 1} (B) {0, –1}
(C) {0, 1, –1} (D) none of these

22. Let f(x) be defined for all $x\in R$ and the continuous. Let $f(x+y)-f(x-y)=4xy\ \forall\ x, y = \in R$ and $f(0)=0$ then

(A) f(x)is bounded
(B) $f(x)+f\left(\frac{1}{x}\right)=f\left(x+\frac{1}{x}\right)+2$
(C) $f(x)+f\left(\frac{1}{x}\right)=f\left(x-\frac{1}{x}\right)+2$
(D) none of these

23. If $g'(x)$ exists for all x, $g'(0)=2$ and $g(x+y)=e^y g(x)+e^x g(y)\ \forall\ x, y$. Then

(A) $g(2x)=-2e^x g(x)$ (B) $g'(x)=g(x)+2e^x$
(C) $\lim_{h\to 0} g(h)/h=3$ (D) None of these

24. If $y=|1-|2-|3-|4-x||||$; then number of points where y is not differentiable; is

(A) 1 (B) 3
(C) 5 (D) >5

25. $f(x)=\begin{cases}\frac{x}{2x^2+|x|}, & x\neq 0\\ 1, & x=0\end{cases}$ then f(x) is

(A) Continuous but non-differentiable at x = 0
(B) Differentiable at x = 0
(C) Discontinuous at x = 0
(D) None of these

26. Let f(x) = cos x and g(x) =

$g(x)=\begin{cases}\text{minimum } \{f(t): 0\le t\le x\}, & x\in[0,\pi]\\ \sin x-1, & x>\pi\end{cases}$

then

(A) g(x) is discontinuous at x = π
(B) g(x) is continuous for $x\in[0,\infty)$
(C) g(x) is differentiable at x = π
(D) g(x) is differentiable for x $[0,\infty)$

27. If f(x + y + z) = f(x). f(y) . f(z) for all x, y, z and f(2) = 4, f′(0) = 3, then f'(2) equals

(A) 12 (B) 9
(C) 16 (D) 6

28. If $f(x)=\lim_{k\to\infty}(\cos x)^{2k}+\cos^{-1}(\sin x)$; then the points of non-differentiability of f(x) is

(A) x = 1 (B) x = – 1
(C) $x=n\pi(n\in 1)$ (D) none of these

29. If f(x) = Max. {1, (cos x + sin x), (sin x – cos x)} $0\le x\le 5\pi/4$, then

(A) f(x) is not differentiable at x = π/6
(B) f(x) is not differentiable at x = 5π/6
(C) f(x) is continuous for $x\in[0, 5\pi/4]$
(D) None of these

30. Let $f: R\to R$ satisfying $|f(x)|\le x^2\ \forall\ x\in R$, then

(A) 'f' is continuous but non-differentiable at x = 0
(B) 'f' is discontinuous at x = 0
(C) 'f' is differentiable at x = 0
(D) None of these

31. Choose the incorrect statement given that f is differentiable

(A) If f is odd and f′(c) = 3 then f′(–c) = 3
(B) If f is even and f′(c) = 3 then f′(–c) – 3
(C) If f is even then f′(0) = 0
(D) If f is even then f′(0) = 0

32. Consider the function f(x)

$$=\begin{cases}x^3 & \text{if } x<0\\ x^2 & \text{if } 0\le x<1\\ 2x-1 & \text{if } 1\le x<2\\ x^2-2x+3 & \text{if } x\ge 2\end{cases},$$

then f is continuous and differentiable for

(A) $x\in R$ (B) $x\in R-\{0,2\}$
(C) $x\in R-\{2\}$ (D) $x\in R-\{1,2\}$

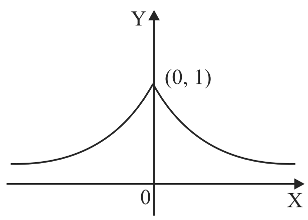
33. The number of points on [0, 2] where

$f(x)=\begin{cases} x\{x\}+1 & 0\le x<1 \\ 2-\{x\} & 1\le x\le 2 \end{cases}$

fails to be continuous or derivable is

(A) 0 (B) 1
(C) 2 (D) 3

34. Which one of the following functions best represent the graph as shown adjacent ?



(A) $f(x)=\dfrac{1}{1+x^2}$ (B) $f(x)=\dfrac{1}{1+\sqrt{|x|}}$
(C) $f(x)=e^{-|x|}$ (D) $f(x)=a^{|x|}\ (a>1)$

35. The number of points where the function $f(x)=(x^2-1)\,|x^2-x-2|+\sin(|x|)$ is not differentiable is

(A) 0 (B) 1
(C) 2 (D) 3

36. Let f(x) be differentiable at x = h then

$\lim\limits_{x\to h}\dfrac{(x+h)\ f(x)\ -\ 2h\ \ f(h)}{x\ -\ h}$ is equal to

(A) $f(h)+2hf'(h)$ (B) $2\,f(h)+hf'(h)$
(C) $hf(h)+2f'(h)$ (D) $hf(h)-2f'(h)$

37. The function $f(x)=\text{maximum}\left\{\sqrt{x(2-x)},2-x\right\}$ is non-differentiable at x equal to

(A) 1 (B) 0, 2
(C) 0, 1 (D) 1, 2

38. Let $f(x)=\begin{cases}\dfrac{\sin|x^2-5x+6|}{x^2-5x+6}, & x\ne 2,3 \\ 1, & x=2 \text{ or } 3\end{cases}$.

The set of all points where f is differentiable is

(A) (∞,∞) (B) $(-\infty,\infty)-\{2\}$
(C) $(-\infty,\infty)-\{3\}$ (D) $(-\infty,\infty)-\{2,3\}$

39. If $f'(a)=\dfrac{1}{4}$, then $\lim\limits_{h\to 0}\dfrac{f(a+2h^2)-f(a-2h^2)}{f(a+h^3-h^2)-f(a-h^3+h^2)}=$

(A) 0 (B) 1
(C) –2 (D) none

40. If $f(x)=\begin{cases}[x]+\sqrt{\{x\}} & x<1 \\ \dfrac{1}{[x]+\{x\}^2} & x\ge 1\end{cases}$, then [where [.] and { . } represents greatest integer part and fractional part respectively.]

(A) f(x) is continuous at x = 1 but not differentiable
(B) f(x) is not continuous at x = 1
(C) f(x) is differentiable at x = 1
(D) $\lim\limits_{x\to 1} f(x)$ does not exist

41. If f(x) has isolated point discontinuity at x = a such that |f(x)| is continuous at x = a then

(A) |f(x)| must be differentiable at x = a
(B) $\lim\limits_{x\to a} f(x)$ does not exist
(C) $\lim\limits_{x\to a} f(x)+f(a)=0$
(D) f(a) = 0

42. Let f be a differentiable function on the open interval (a, b). Which of the following statements must be true?

I. f is continuous on the closed interval [a, b]
II. f is bounded on the open interval (a, b)
III. If $a<a_1<b_1<b$, and $f(a_1)<0<f(b_1)$, then there is a number c such that $a_1<c<b_1$ and $f(x)=0$

(A) I and II only (B) I and III only
(C) II and III only (D) only III

43. Let $f:R\to R$, $f(x-f(y))=f(f(y))+xf(y)+f(x)-1\ \forall$ x, y ∈ R, if f(0) = 1 and f′(0) = 0, then

(A) $f(x)=1-\dfrac{x^2}{2}$ (B) $f(x)=x^2+1$
(C) $f(x)=\left(\dfrac{2x+1}{x+1}\right)$ (D) none of these

44. If f : [–2a, 2a] → R is an odd function such that f(x) = f(2a – x) for x ∈ (a, 2a). if the left hand derivative of f(x) at x = a is zero, then the left hand derivative of f(x) at x = –a is

(A) 1 (B) –1
(C) 0 (D) none

45. Suppose that the differentiable functions u, v, f, g : R → R satisfy $\lim\limits_{x\to\infty} u(x)=2$, $\lim\limits_{x\to\infty} v(x)=3$, $\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to\infty} g(x)=\infty$ and $\dfrac{f'(x)}{g'(x)}+u(x)\dfrac{f(x)}{g(x)}=(vx)$

then $\lim_{x\to\infty}\frac{f(x)}{g(x)}$ is equal to (given that it exists)

(A) 1 (B) 1/2
(C) 2 (D) None

46. Suppose f is a differentiable function such that $f(x+y)=f(x)+f(y)+5xy$ for all x, y and $f'(0)=3$. The minimum value of $f(x)$ is

(A) -1 (B) $-9/10$
(C) $-9/25$ (D) None

47. Let h(x) be differentiable for all x and let $f(x)=(kx+e^x)h(x)$ where k is some constant. If $h(0)=5$, $h'(0)=-2$ and $f'(0)=18$ then the value of k is equal to

(A) 5 (B) 4
(C) 3 (D) 2.2

48. If for a function f(x) : f(2) = 3, f′(2) = 4, then $\lim_{x\to2}[f(x)]$, where [.] denotes the greatest integer function, is

(A) 2 (B) 3
(C) 4 (D) dne

49. If $f(x)=\begin{cases}[x]+\sqrt{\{x\}} & x<1\\ \frac{1}{[x]+\{x\}^2} & x\ge1\end{cases}$, then

[where [.] and { . } represent greatest integer and fractional part functions respectively]

(A) f(x) = is continuous at x = 1 but not differentiable
(B) f(x) is not continuous at x = 1
(C) f(x) is differentiable at x = 1
(D) $\lim_{x\to1} f(x)$ does not exits

50. Let $f''(x)$ be continuous at x = 0 and $f''(0)=4$ then value of $\lim_{x\to0}\frac{2f(x)-3f(2x)+f(4x)}{x^2}$ is

(A) 11 (B) 2
(C) 12 (D) none

MULTIPLE CORRECT ANSWER TYPE FOR JEE ADVANCED

51. Let $f(x)=|x-1|([x]-[-x])$, then which of the following statement(s) is/are correct?
(where [x] denotes greatest integer function)

(A) f(x) is continuous at x = 1
(B) f(x) is derivable at x = 1
(C) f(x) is non-derivable at x = 1
(D) f(x) is discontinuous at x = 1.

52. Which of the following function will have tangent at indicated point

(A) $f(x)=\begin{cases}x^2\sin\frac{1}{x} & , x\ne0\\ 0 & , x=0\end{cases}$ at x = 0
(B) f(x) = sgn (sin x) at x = π
(C) f(x) = sgn (x^2) at x = 0
(D) $f(x)=|x-1|^{11/10}$ at x = 1

53. Let f(x) = sin πx, g(x) = sgn (x) and h(x) = gof(x) then

(A) h(x) is discontinuous at infinite number of points
(B) $h'(x)=0$ for all $x\in R-I$
(C) $\lim_{x\to I}h(x)$ does not exist
(D) h(x) is periodic with period 1.

54. If $f(x)=\frac{\tan[x]\pi}{[1+|\log(\sin^2x+1)|]}$, where [.] denotes the greatest integer function then f(x) is

(A) discontinuous ∀ x ∈ I
(B) continuous ∀ x ∈ R
(C) non differentiable ∀ x ∈ I
(D) a periodic function with no fundamental period

55. Let f and g be two functions defined as follows :

$f(x)=\frac{x+|x|}{2}$ for all x & $g(x)=\begin{cases}x & \text{for } x<0\\ x^2 & \text{for } x\ge0\end{cases}$ then

(A) (gof)(x) & (fog)(x) are both continuous for all x ∈ R
(B) (gof)(x) & (fog)(x) are unequal functions
(C) (gof) is differentiable at x = 0
(D) (fog)(x) is not differentiable at x = 0.

56. If a function f : R → R satisfies $|\Sigma 3^k(f(x+ky)-f(x-ky))|\le1$ for every n ∈ N and for all x y ∈ R, then

(A) $|3^n(f(x+ny)-f(x-ny))|\le2$
(B) $|f(u)-f(v)|\le\frac{2}{3^n}$
(C) f(x) is a constant function
(D) $f'(x)=1$ ∀ x ∈ R

57. If we define new derivative of function f to be

$\Delta(f(x))=\lim_{\Delta x\to0}\frac{f^3(x+\Delta x)-f^3(x)}{\Delta x}$, then

(A) $\Delta(x) = 3x^3$
(B) $\Delta(\tan x)$ at $x = \frac{\pi}{4}$ equals 6
(C) $\Delta(\sin x)$ at $x = 0$ equals 1
(D) $\Delta(\cos^{-1} x)$ at $x = 0$ equal $-\frac{3\pi^2}{4}$

58. if $f(x) = \text{minimum}\left(\cos x, \frac{1}{2}, \{\sin x\}\right), 0 \le x \le 2\pi$, where { . } represents fractional part function, then f(x) is differentiable at
(A) $\frac{3\pi}{2}$ (B) $\frac{\pi}{4}$
(C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

59. If $f(x) = \text{minimum}\{x^3 - 1, -x + 1, \text{sgn}(-x)\}$, then f(x) is
(A) continuous at x = 0
(B) differentiable at x = 0
(C) continuous at x = 2
(D) differentiable at x = 2

60. If [.] denotes greatest integer function, then for the function $y = \sin \pi (|x| + [x])$, which of the following is correct
(A) not continuous at x = –1/2
(B) continuous at x = 0
(C) differentiable in [–1, 0)
(D) differentiable in (0,1]

61. If f(x) is differentiable everywhere and f(a) = 0, then at x = a
(A) | f | is differentiable
(B) $|f|^2$ is differentiable
(C) f | f | is differentiable
(D) $f + |f|^2$ is differentiable

62. Let $f(x) = x^2 \sin\frac{1}{x}$ for $0 < x \le 1$ and f (0) = 0. If $g(x) = x^2$ for $x \in [0, 1]$ then which of the following statement(s) is/are correct?
(A) f (x) is differentiable in [0, 1]
(B) $\lim_{x\to 0} \frac{f(x)}{g(x)}$ does not exist.
(C) $\lim_{x\to 0} \frac{f(x)}{g(x)}$ does not exist
(D) $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ does not exist.

63. If $f(x) = \begin{cases} x+1 & , \quad x \ge 1 \\ 2-x & , \quad x < 1 \end{cases}$ and $g(x) = f(x) + f(|x|)$, then g(x) is
(A) not continuous at x = –1
(B) not continuous at x = 1
(C) continuous at x = 0
(D) not differentiable at x = 0

64. A function f(x) satisfies the relation
$f(x + y) = f(x) + f(y) + xy(x + y) \ \forall\ x, y \in R$.
If $f'(0) = -1$, then
(A) f(x) is a polynomial function
(B) f(x) is an exponential function
(C) f(x) is twice differentiable for all $x \in R$
(D) $f'(3) = 8$

65. Function $f(x) = \max(|\tan x|, \cos|x|)$ is
(A) not differentiable at 4 points in $(-\pi, \pi)$
(B) discontinuous at 2 points in $(-\pi, \pi)$
(C) not differentiable at only 2 points in $(-\pi, \pi)$
(D) in $(-\pi, \pi)$ there are only 2 points where f(x) is continuous but not differentiable

66. Let a differentiable function f(x) be such that $|f(y) - f(y)| \le \frac{1}{2}|x - y| \ \forall\ x, y \in R$ and $f'(x) \ge \frac{1}{2}$. Then the number of points of intersection of the graph of y = f(x) with
(A) the line y = x is one
(B) the curve $y = -x^3$ is one
(C) the curve 2y = |x| is three
(D) None of these

67. Given that the derivative f′(a) exists. State which of the following statements are true
(A) $f'(a) = \lim_{h\to 0} \frac{f(h) - f(a)}{h - a}$
(B) $f'(a) = \lim_{h\to 0} \frac{f(a) - f(a-h)}{h}$
(C) $f'(a) = \lim_{t\to 0} \frac{f(a+2t) - f(a)}{t}$
(D) $f'(a) = \lim_{t\to 0} \frac{f(a+2t) - f(a+t)}{2t}$

68. Let $f(x)=\begin{cases}\sin^2 x.\cos\frac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$

(A) f is continuous at $x = 0$
(B) $f'(0)=1$
(C) $f'(0)$ does not exist
(D) f' is not continuous at $x = 0$.

69. Let $f(x) = g'(x)\dfrac{e^{a/x}-e^{-a/x}}{e^{a/x}+e^{-a/x}}$ where g' is the derivative of g and is a continuous function and $a > 0$ then $\lim\limits_{x\to 0} f(x)$ exists if

(A) g(x is polynomial
(B) $g(x)=x$
(C) $g(x)=x^2$
(D) $g(x)=x^3 h(x)$ where h(x) is a polynomial

70. Let $f : R \to R$ be a function such that,
$f\left(\dfrac{x+y}{3}\right) = \dfrac{f(x)+f(y)}{3}$, $f(0)=3$ and $f'(0)=3$, then

(A) $\dfrac{f(x)}{x}$ is differentiable in R
(B) $f'(x)=3$
(C) f(x) is continuous in R
(D) f(x) is bounded in R

Assertion (A) and Reason (R)

(A) Both A and R are true and R is the correct explanation of A.
(B) Both A and R are true but R is not the correct explanation of A.
(C) A is true, R is false.
(D) A is false, R is true.

71. Consider the function $f(x)=x^2-2x$ and $g(x)=-|x|$
Assertion (A) : The composite function $F(x) = f(g(x))$ is not derivable at $x = 0$.
Reason (R) : $F'(0^+)=2$ and $F'(0^-)=-2$.

72. **Assertion (A) :** There is no polynomial function f such that $f(x+y)=f(x)+yf(f(x))$ for all $x, y\in R$
Reason (R) : $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=f(f(x))$.
If f is of degree n then the equation $n-1 = n^2$ has no positive integral solution.

73. **Assertion (A) :** There are exactly three functions $f : R \to R$ such that
$f(x) f(y) = f(x) + f(y) - f(xy)$ ...(1)
$1 + f(x + y) = f(xy) + f(x) f(y)$ ...(2)
Reason (R) : Adding (1) and (2), we get
$1 + f(x + y) = f(x) + f(y)$.

Now $f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0}\dfrac{f(h)-1}{h} = f'(0)$.
$\Rightarrow$ $f(x) = ax + b$ where a & b can be determined by using function's value at $x = 0, 1$.

74. **Assertion (A) :** The function $y = \sin^{-1}(\cos x)$ is not differentiable at $x = n\pi$, $n \in I$ is particular at $= \pi$.
Reason (R) : $\dfrac{dy}{dx}=\dfrac{-\sin x}{|\sin x|}$ so the function is not differentiable at the points where $\sin x = 0$.

75. Consider two functions $f(x) = \sin x$ and $g(x) = |f(x)|$.
Assertion (A) : The function $h(x) = f(x) g(x)$ is not differentiable in $[0, 2\pi]$.
Reason (R) : f (x) is differentiable and g (x) is not differentiable in $[0, 2\pi]$

76. **Assertion (A) :** The function
$f(x)= c_1e^{|x|}+c_2|x|^3$, where c_1, c_2 are constants is differentiable at $x = 0$, provided $c_1 = 0$.
Reason (R) : $e^{|x|}$ is not differentiable at $x = 0$.

77. Consider two functions $f(x) = \sin x$ and $g(x) = |f(x)|$.
Assertion (A) : The function $h(x) = f(x) g(x)$ is not differentiable in $[0, 2\pi]$
Reason (R) : f (x) is differentiable and g (x) is not differentiable in $[0, 2\pi]$

78. Consider the function $f(x)=\cot^{-1}\left(\operatorname{sgn}\left(\dfrac{[x]}{2x-[x]}\right)\right)$
where [ ] denotes the greatest integer function
Assertion (A) : f(x) is discontinuous at $x = 1$.
Reason (R) : f(x) is non derivable at $x = 1$.

79. Let $f(x) = \operatorname{sgn} x$ and $g(x)=\begin{cases}x-1 & 0<x\le 2\\ 1+x^2 & 2<x\le 4\end{cases}$ then
Assertion (A): The function (fog)(x) is differentiable at $x = 2$.
Reason (R) : If (fog) (x) is differentiable at $x = a$ then g(x) is differentiable at $x = a$ and g(x) is differentiable at $x = g(a)$.

80. Assertion (A) :
$f(x) = \left|(x-1)^{2p_1+1}(x-2)^{2p_2+1}....(x-n)^{2p_n+1}\right|$ is differentiable everywhere where $p_1, p_2, ...p_n \in N$.
Reason (R) : $f(x) = x|x|$ is differentiable everywhere.

Comprehension - 1

$$\text{Let } f(x) = \begin{cases} e^{\{x^2\}} - 1, & x > 0 \\ \dfrac{\sin x - \tan x + \cos x - 1}{2x^2 + \ln(2+x) + \tan x}, & x < 0 \\ 0, & x = 0 \end{cases}$$

where { } represents fractional part function. Lines L_1 and L_2 represent tangent and normal to curve $y = f(x)$ at $x = 0$. Consider the family of circles touching both the lines L_1 and L_2.

81. Ratio of radii of two circles belonging to this family cutting each other orthogonally is
(A) $2+\sqrt{3}$ (B) $\sqrt{3}$
(C) $2+\sqrt{2}$ (D) $2-\sqrt{2}$

82. A circle having radius unity is inscribed in the triangle formed by L_1 and L_2 and a tangent to it. Then the minimum area of the triangle possible is
(A) $3+\sqrt{2}$ (B) $3-\sqrt{2}$
(C) $3+2\sqrt{2}$ (D) $3-2\sqrt{2}$

83. If centres of circles belonging to family having equal radii 'r' are joined, the area of figure formed is
(A) $2r^2$ (B) $4r^2$
(C) $8r^2$ (D) r^2

Comprehension - 2

Let f(x) be a real valued function not identically zero, which satisfies the following conditions.

I. $f(x+y^{2n+1}) = f(x) + \{f(y)\}^{2n+1}$, $n \in N$, x, y are any real numbers

II. $f'(0) \geq 0$.

84. The value of $f'(10)$ is
(A) 10 (B) 0
(C) $2n+1$ (D) 1

85. The function f(x) is
(A) odd
(B) even
(C) neither even nor odd
(D) even as well as odd

86. The function f(x) is
(A) Continuous and differentiable everywhere
(B) Continuous everywhere but not differentiable at some points
(C) Discontinuous at exactly one point
(D) Discontinuous at infinitely many points

Comprehension - 3

Suppose f, g and h be three real valued function defined on R.

Let $f(x) = 2x + |x|$, $g(x) = \frac{1}{3}(2x - |x|)$ and $h(x) = f(g(x))$

87. The range of the function

$k(x) = 1 + \frac{1}{\pi}(\cos^{-1}(h(x)) + \cot^{-1}(h(x)))$ is equal to

(A) $\left[\frac{1}{4}, \frac{7}{4}\right]$ (B) $\left[\frac{5}{4}, \frac{11}{4}\right]$
(C) $\left[\frac{1}{4}, \frac{5}{4}\right]$ (D) $\left[\frac{7}{4}, \frac{11}{4}\right]$

88. The domain of definition of the function $l(x) = \sin^{-1}(f(x)) - g(x))$ is equal to
(A) $\left[\frac{3}{8}, \infty\right)$ (B) $(-\infty, 1]$
(C) $[-1, 1]$ (D) $\left(-\infty, \frac{3}{8}\right]$

89. The function $T(x) = f(g(f(x))) + g(f(g(x)))$, is
(A) continuous and differentiable in $(-\infty, \infty)$.
(B) continuous but not derivable $\forall x \in R$.
(C) neither continuous nor derivable $\forall x \in R$.
(D) an odd function.

Comprehension - 4

A function f (x) is said to be differentiable at x = c, if $f'(c^+) = f'(c^-)$ = a finite quantity. Also f (x) is said to be bounded if $|f(x)| < n\ \forall x$ for which f (x) is defined (where n is finite)

90. Let $f : R \rightarrow R$ and $c \in R$ such that
$\lim_{n\to\infty} n\left(f\left(c+\frac{1}{n}\right) - f(c)\right) = a\ (a \in R)$ then
(A) $f'(c)$ exists and is equal to a
(B) $f'(c)$ does not exists
(C) $f'(c)$ exists but is not equal to a
(D) $f'(c)$ may or may not exist.

91. If $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$ and f(x) be continuous on R. Then

(A) f (x) is bounded on R and attains both maximum and minimum on R.
(B) f (x) is unbounded on R but attains minimum on R.
(C) f (x) is bounded on R and attains either maximum or minimum on R.
(D) f is unbounded.

92. f (x) is defined on [0, 1] and

$f(x)=\begin{cases} x^a \sin(x^{-c}) & x \neq 0 \\ 0 & x = 0 \end{cases}$ where $a, c \in R$ and $c > 0$

(A) $f'(0)$ exists if $0 < a < 1$
(B) f (x) is continuous if $a < 0$
(C) $f'(x)$ is bounded and continuous if $a > c$
(D) f (x) and $f'(x)$ are continuous if $a > 1 + c$.

Comprehension - 5

Let $f : R \to R$ be a differential function satisfying $f\left(\frac{x+y}{3}\right) = \frac{2+f(x)+f(y)}{3}$ for all real x and y and $f'(2) = 2$.

93. The range of f(|x|) is
(A) $[0, \infty)$ (B) $[1, \infty)$
(C) $[2, \infty)$ (D) None of these

94. If $f(x) = |f(|x|) - 3|$ for all $x \in R$, then for g(x)
(A) one non-differentiable point
(B) two non-differentiable point
(C) three non-differentiable point
(D) four non-differentiable point

95. The number of solutions of the equation $x^2 + (f(|x|)^2) = 9$ are
(A) 0 (B) 2
(C) 3 (D) 5

MATCH THE COLUMNS FOR JEE ADVANCED

96. **Column-I** **Column-II**

(A) Let $f(x) = \begin{cases} \tan^{-1} x, & |x| \geq 1 \\ \frac{x^2-1}{4}, & |x| < 1 \end{cases}$ $\frac{x^2-1}{4}$, $|x| < 1$, then f(x) is not differentiable at x equal to — (P) –1

(B) $f(x) = (x^2 - 4)\,|x^2 - 5x + 6| + \cos|x|$ is non derivable at x equal to — (Q) 1

(C) If $\sin(x + y) = e^{x+y} - 2$, then $\frac{dy}{dx}$ is equal to — (R) 2

(D) Let f: $R \to R$ is defined by the equation — (S) 3
$f(x + y) = f(x) f(y)\ \forall x, y \in R$, $f(0) \neq 0$ and $f'(0) = 2$
then $\frac{f'(x)}{f(x)}$ is equal to — (T) None of the above values

97. **Column - I** **Column - II**

(A) Let $g(x) = \begin{cases} a\sqrt{x+1} & \text{if } 0 < x < 3 \\ bx + 2 & \text{if } 3 \leq x < 5 \end{cases}$, if g(x) is differentiable on (0, 5) then (a + b) equals — (P) 0

(B) Let a and b be real numbers and let $f(x) = a \sin x + b\sqrt[3]{x} + 4, \forall x \in R$ — (Q) 1
If $f(\log_{10}(\log_3 10)) = 5$ then the value of $f(\log_{10}(\log_{10} 3))$ is equal to

(C) $\lim\limits_{x \to 0} \frac{\sin^4 x + \sin^4 2x + \sin^4 3x}{\sin^3 x + \sin^3 2x + \sin^3 3x}$ equals — (R) 2

(D) Number of solution(s) of the equation $2^{\log_{10} x} + 8 = (x-8)^{\frac{1}{\log_{10} 2}}$ is — (S) 3

(T) 4

98.

Column – I	Column – II
(A) If f(xy) = f(x) . f(y) and f is differentiable at x = 1 such that $f'(1) = 1$ also $f(1) \neq 0$, then $f'(7)$ equals	(P) 1
(B) If [.] denotes greatest integer function, then number of points at which the function $f(x) = \|x^2 - 3x + 2\| + \|\sin x\| - [x - 1/2]$, $-\pi \leq x \leq \pi$, is non differentiable, is	(Q) 13
(C) Let $f(x) = [a + 7 \sin x]$, $x \in (0, \pi)$, $a \in I$, [.] denotes greatest integer function. Then number of points at which f(x) is not differentiable is	(R) 5
(D) If for a continuous function, f, $f(0) = f(1) = 0$, $f'(1) = 2$ and $y(x) = f(e^x)\, e^{f(x)}$, then $y'(0)$ is equal to	(S) 3
	(T) 2

99.

Column-I	Column-II
(A) If $f(x) = \begin{cases} \dfrac{\left(\dfrac{\pi}{2} - \sin^{-1}(1-\{x\}^2)\right)\sin^{-1}(1-\{x\})}{\sqrt{2}\,(\{x\}-\{x\}^3)}, & x > 0 \\ k, & x = 0 \\ \dfrac{A\sin^{-1}(1-\{x\})\cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}\,(1-\{x\})}, & x < 0 \end{cases}$ is continuous at x = 0, then the value of $\sin^2 k + \cos^2\left(\dfrac{A\pi}{\sqrt{2}}\right)$, is (where {x} denotes fractional part function)	(P) 2
(B) If $f(x) = [2 + 5\,\|n\|\sin x]$ where $n \in I$ has exactly 19 points of non-differentiability in $(0, \pi)$, then possible value(s) of 'n' are (where [x] denotes greatest integer function)	(Q) –2
(C) Let $f(x) = \begin{cases} x\,\dfrac{\left(\frac{3}{4}\right)^{\frac{1}{x}} - \left(\frac{3}{4}\right)^{\frac{-1}{x}}}{\left(\frac{3}{4}\right)^{\frac{1}{x}} + \left(\frac{3}{4}\right)^{\frac{-1}{x}}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. If $P = f'(0^-) - f'(0^+)$, then $\lim\limits_{x \to P^-} \dfrac{(\exp((x+2)\ln 4))^{\frac{[x+1]}{4}} - 16}{4^x - 16}$ is less than (where [x] denotes greatest integer function)	(R) 3
(D) The number of points at which $g(x) = \dfrac{1}{1 + \dfrac{2}{f(x)}}$ is not differentiable where $f(x) = \dfrac{1}{1 + \dfrac{1}{x}}$, is	(S) –3
	(T) 5

100.

	Column I	Column – II
(A)	If $f:[0,1]\to[0,1]$ is continuous, then the number of roots of the equation $f(x)=x^3$ is	(P) one
(B)	If f is derivable, then between the consecutive roots of $f'(x)=0$, the number of roots of $f(x)=0$ is	(Q) at least one
(C)	The number of values of x at which $f(x)=\|1-\|x-1\|\|$ is not differentiable	(R) at most one
(D)	Let $f(x)=\begin{cases} x, & \text{if } x \text{ is rational} \\ 2-x, & \text{if } x \text{ is irrational}\end{cases}$ The number of values of x at which f(x) is continuous is	(S) more than one

Review Exercises for JEE Advanced

1. Let $f(x)=\dfrac{\sqrt{x+1}-1}{\sqrt{x}}$ for $x\neq 0$, $f(0)=0$. Is the function f(x) continuous and differentiable at $x=0$?

2. Let g(x) be polynomial of degree atmost two and f(x) be continuous at x = 0 and differentiable at x = 1. Find

$$g(x) \text{ if } f(x)=\begin{cases} \dfrac{5xe^{1/x}+2x}{3-e^{1/x}}, & x<0 \\ g(x), & 0\le x\le 1 \\ (x-1)^2\sin\dfrac{1}{(1-x)^2}+x, & x>1 \end{cases}.$$

3. Let $f(x)=\begin{cases} -x-1, & -2\le x<0 \\ x^2-x, & 0\le x<2 \\ 2-x, & 2\le x\le 3\end{cases}$ and

$g(x)=f(|x|)+|f(x)|$ check continuity and differentiability of g(x) over [–2, 3].

4. Check the continuity and differentiability of $f(x)=\sin^{-1}\left(\dfrac{2\sqrt{x^2-1}}{x^2}\right)$.

5. Let $f(x)=(x^2-4)\,|(x^3-6x^2+11x-6)|+\dfrac{x}{1+|x|}$. Find the set of points at which the function f(x) is not differentiable.

6. Examine the differentiability of

$f(x)=\sqrt{x+2\sqrt{2x-4}}+\sqrt{x-2\sqrt{2x-4}}$ over its domain.

7. Let $f(x)=x^3-3x$ and

$$g(x)=\begin{cases}\min(f(t):0\le t\le x), & 0\le x\le 2 \\ 2x-5, & 2<x\le 3 \\ (x-2)^2, & x>3\end{cases}.$$

Draw g(x) and discuss continuity and differentiability of g in (0, 4).

8. Let $f(x)=\begin{cases}x^2+a, & 0\le x<1 \\ 2x+b, & 1\le x\le 2\end{cases}$

and $g(x)=\begin{cases}3x+b, & 0\le x<1 \\ x^3, & 1\le x\le 2\end{cases}$. If $\dfrac{d(f(x))}{d(g(x))}$ exists at x = 1, find the values of a, b and also its value.

9. If f and g be two functions having the same domain D, if f and g be derivable at $x_0\in D$, and if $f(x_0)\neq g(x_0)$, then prove that each of the function max. {f, g} and min. {f, g} is derivable at x_0. What happens if $f(x_0)=g(x_0)$?

10. (i) $f:[0,1]\to[1,2]$ be a differentiable function, then prove that the number of solution of $f(x)-e^{x(x-1)}-\log_2(1+x)=0$ will be at least one.

(ii) Let $f:[1,e]\to[0,1]$ be continuous then prove that $f(x)=\ell n\, x$ has atleast one solution in [1, e].

11. If a differentiable function f(x) satisfies the relation

$$\frac{x}{y}f\left(\frac{x}{y}\right)=x\,f(x)-yf(y)\ \forall\ x,y\in R\ \&\ f(e)=1/e,$$

then prove that $f(x)=\dfrac{\log x}{x}$.

12. Given a real valued function f(x) as follows :

$f(x) = \dfrac{x^2+2\cos x-1}{x^4}$ for $x < 0$; $f(0) = \dfrac{1}{12}$ &

$f(x) = \dfrac{\sin x - \ell n(e^x \cos x)}{6x^2}$ for x >. Test the continuity and differentiability of f(x) at x = 0.

13. Let f be given by

$$f(x)=\begin{cases} x^2\sin(x^{-4/3}) & \text{when } x\le 0 \\ 0 & \text{when } x=0 \end{cases}.$$

Show that f has a derivative for all values of x, and that $f'(0) = 0$. Prove that f' is not, however, continuous at x = 0

14. Draw the graph of the function $f(x) = x - |x - x^2|$, $-1 \le x \le 1$ and discuss its continuity in [–1, 1] and differentiability in (–1, 1).

15. If $f(x) = \begin{cases} \dfrac{\tan[x^2]\pi}{ax^2}+ax^3+b & , \ 0\le x\le 1 \\ 2\cos \pi x+\tan^{-1}x & , \ 1<x\le 2 \end{cases}$ is differentiable in [0, 2], then $a = \dfrac{1}{k_1}$ and $b = \dfrac{\pi}{4} - \dfrac{26}{k_2}$. Find $k_1^2 + k_2^2$ { where [ ] denotes greatest integer function}.

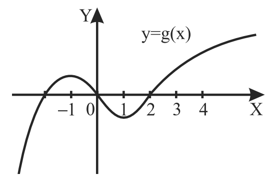
16. If $f(x) = \begin{cases} x^a \cdot \cos\left(\dfrac{1}{x^3}\right), & x \ne 0 \\ 0, & x = 0 \end{cases}$ is continuous but non-differentiable at x = 0, then all possible values of a.

17. Examine the function, $f(x) = x \cdot \dfrac{a^{1/x} - a^{-1/x}}{a^{1/x} + a^{-1/x}}$, $x \ne 0$ $(a > 0)$ and $f(0) = 0$ for continuity and existence of the derivative at the origin .

18. Let $f(x) = x^4 - 8x^3 + 22x^2 - 24$ and

$g(x) = \begin{cases} \min f(t) \ ; x \le t \le x+1, -1 \le x \le 1 \\ x - 10 \ ; x > 1 \end{cases}$ Discuss the differentiability of g(x) in [–1, 2].

19. For the function g whose graph is given, arrange the following numbers in increasing order.
0, g′(–2), g′(0), g′(2), g′(4)



20. Let $f(x) = \max.\{|x^2 - 2|x||, |x|\}$ and $g(x) = \min.\{|x^2 - 2|x||, |x|\}$ then show that f(x) is not differentiable at 5 points and g (x) is non differentiable at 7 points.

21. Consider function $f : R - \{-1, 1\} \to R$.
$f(x) = \dfrac{x}{1-|x|}$. Find f′(x), draw the graph of f and prove that it is not derivable at the origin.

22. Suppose the function f satisfies the conditions :
(i) $f(x + y) = f(x)f(y)$ for all x and y.
(ii) $f(x) = 1 + x.g(x)$ where $\lim_{x\to 0} g(x) = 1$
Show that the derivative f′(x) exists and f′(x) = f(x) for all x.

23. If $f(x) = |x - 1| . ([x] - [-x])$, then find $f'(1^+)$ and $f'(1^-)$ where [x] denotes greatest integer function.

24. Let $f : R \to R$ be continuous function $|f(x)| \le x^{4n}$, $n \in N$. Prove that f(x) is differentiable at x = 0.

25. Find the values of

(a) $\lim_{h\to 0} \dfrac{f(3+h)^2 - f(3-h^2)}{2h^2}$ if $f'(3) = 2$;

(b) $\lim_{h\to 0} \dfrac{f(a+2h^2) - f(a-2h^2)}{h^2}$ if $f'(a) = \dfrac{1}{4}$.

26. If $f(x) = \begin{cases} \dfrac{5e^{1/x}+2}{3-e^{1/x}}, & x \ne 0 \\ 0 & , x \ne 0 \end{cases}$ Examine f(x), xf(x) and $x^2 f(x)$ for continuity and differentiability at x = 0

27. $f(x) = \begin{bmatrix} 1-x, (0 \le x \le 1) \\ x+2, (1 < x < 2) \\ 4-x, (2 \le x \le 4) \end{bmatrix}$. Discuss the continuity & differentiability of $y = f(f(x))$ for $0 \le x \le 4$.

28. The function $f(x) = \begin{cases} ax(x-1)+b, & x<1 \\ x-1 & , 1\le x\le 3 \\ px^2+qx+2, & x>3 \end{cases}$ find the values of the constant a, b p and q so that
(i) f(x) is continuous for all x.
(ii) f′(1) does not exist.
(iii) f′(x) is continuous at x = 3.

29. Discuss the continuity on $0 \le x \le 1$ & differentiability at x = 0 for the function f(x) = x.

$$\sin\frac{1}{x}\ \sin\left(\frac{1}{x.\sin 1/x}\right) \text{ where } x \ne 0,\ x \ne 1/r\pi \ \&$$

$f(0) = f(1/r\pi) = 0, r = 1, 2, 3, ...$

30. Suppose that instead of the usual definition of the derivative Df(x), we define a new of derivative, D*f(x), by the formula,

$$D^*(x) = \lim_{h\to 0} \frac{f^2(x+h) - f^2(x)}{h},$$

where $f^2(x)$ means $[f(x)]^2$. Derive the formula for computing the derivative D* of a product & quotient.

31. Let $f(x+y) - 2f(x-y) + f(x) - 2f(y) = y - 2$ for all real x and y. If f'(x) exist, find f(x).

32. Let $f(xy) = xf(x) + yf(y)$ for all real x and y. If f is a differentiable function and f(0) = 0, find f(x).

33. A differentiable function f satisfies the relation $f(xyz) = f(x) + f(y) + f(z)\ \forall\ x, y, z \in R^+$ and $f'(1) = 1$. Find f(x).

34. A differentiable function f satisfies the relation $xf(y) + yf(x) = (x + y)\ f(x)\ .\ f(y)\ \ \forall\ x, y \in R$. If $f'(0) = 0$, find f(x).

35. A differentiable function f satisfies the relation $f(x + y) = f(x) + f(y) + f(x).f(y)\ \forall\ x, y \in R$. If $f(0) \ne -1$ and $f'(0) = -1$, find f(x).

Target Exercises for JEE Advanced

1. Let 'f' be a function whose graph is obtained by summing the ordinate values of the graph of another function 'g' after shifting its graph by 1 unit leftward and rightward.

If $g(x) = \begin{cases} 1-|x| & , |x| \le 1 \\ 0 & , |x| > 1 \end{cases}$ discuss the continuity and differentiability of f(x).

2. Check the differentiability of

$$f(x) = \begin{cases} (x-e)2^{-2^{\left(\frac{1}{e-x}\right)}}, & x \ne e \\ 0 & , x = e \end{cases} \text{ at } x = e.$$

3. Let $f(x) = \begin{cases} \sqrt{x}\left(1 + x\sin\frac{1}{x}\right) & , x > 0 \\ -\sqrt{|x|}\left(1 + x\cos\frac{1}{x}\right) & , x < 0 \\ 0 & , x = 0 \end{cases}$.

Show that f(x) is not differentiable at x = 0.

4. A differentiable function f satisfies the relation $f(x^2 + y^2) = (f(x))^2 + (f(y))^2\ \forall\ x, y \in R$. Find f(x).

5. If $f(x) + f(y) = f\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)\ \forall\ x, y \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ and $\lim_{x\to 0}\frac{f(x)}{x} = 1$, find f(x).

6. Let $f(x) = x^3 - x^2 + x + 1$ and $g(x) = 3 - x$

We define a function

$$h(x) = \begin{cases} \max\{f(u)\ ,\ 0 \le u \le x\}\ , 0 \le x \le 1 \\ \min\{g(v)\ ,\ 1 < v \le x\}\ , 1 < x \le 2 \end{cases}$$

Discuss the continuity and differentiability of h(x) in the interval [0, 2].

7. If $g(x) = \begin{cases} 0 & \text{for } -e \le x < 1 \\ \left\{1 + \frac{1}{3}\sin(\ln x^{2\pi})\right\} & \text{for } 1 \le x \le e \end{cases}$.

where { } denotes the fractional part function.

$$f(x) = \begin{cases} xg(x) & \text{for } 1 \le x \le e \\ x(g(x)+1) & \text{otherwise} \end{cases}.$$

Discuss continuity and differentiability of f(x) over its domain.

8. Suppose f is differentiable on the interval (a, ∞) and $\lim_{x\to\infty}(f(x) + f'(x)) = \ell\ (\ell \in R)$.

Prove that $\lim_{x\to\infty} f(x) = \ell$ and $\lim_{x\to\infty} f'(x) = 0$.

9. Let f be a continuous and differentiable function in (a, b), $\lim_{x\to a^+} f(x) = \infty$ and $\lim_{x\to b^-} f(x) = -\infty$ and $f'(x) + f^2(x) \ge -1$ for $a < x < b$. Prove that $b - a > \pi$.

10. If

$$f(n) = \lim_{x\to 0} \frac{1 - \cos^2 x \cos^2\frac{x}{2}.\cos^2\frac{x}{8}.......\cos^2\frac{x}{2^{n-1}}}{x^2}$$

and $\lambda = \lim_{x\to\infty} f(n)$. Find the absolute value of 27abc for which the function

$$g(x) = \begin{cases} \frac{a}{\pi}\cot^{-1} x & x < 0 \\ \lambda & x = 0 \\ \frac{b}{\pi}x + c & x > 0 \end{cases}$$

is differentiable at $x = 0$.

11. Let $f(x) = \begin{cases} ax(x-1)+b \; ; & x < 1 \\ x+2 \; ; & 1 \le x \le 3 \\ px^2+qx+2 \; ; & x > 3 \end{cases}$ is continuous for all x except x = 1 but $|f(x)|$ is differentiable every where and f'(x) is continuous at x = 3 and $|a + b + p + q| = \frac{k}{18}$, then find the value of k.

12. Two real valued, continuous and differentiable functions f(x) and g(x) defined on R satisfy the conditions

$f(x+y) = f(x)g(y) + g(x)f(y)$

$g(x+y) = g(x)g(y) - f(x)f(y)$

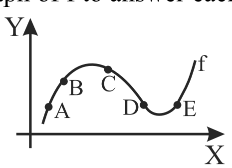
If $g'(0) = 0$, prove that $f^2(x) + g^2(x) = 1$.

13. Sketch the graph of the function

$$f(x) = \begin{cases} \frac{x}{1+|x|} & \text{if } |x| \ge 1 \\ \frac{x}{1-|x|} & \text{if } |x| < 1 \end{cases}$$

and find the domain of $f'(x)$.

14. Use the graph of f to answer each question.



(i) Between which two consecutive points is the average rate of change of the function greatest?

(ii) Is the average rate of change of the function between A and B greater than or less than the instantaneous rate of change at B ?

(iii) Sketch a tangent line to the graph between the points C and D so that the slope of the tangent line is the same as the average rate of change of the function between C and D.

(iv) Give any sets of consecutive points for which the average rates of change of the function are approximately equal.

15. Given $f(x+y+z) = f(x)\,f(y)\,f(z)$ for all real x, y, z if $f(2) = 4$ and $f'(0) = 3$, then show that $f(0) = 1$ and $f'(2) = 12$.

16. Let $f(x) = x^3 - 3x^2 + 6 \;\forall\; x \in R$ and

$$g(x) = \begin{cases} \max.\{f(t),\, x+1 \le t \le x+2\}, & -3 \le x \le 0 \\ 1 - x \text{ for } x \ge 0 \end{cases}.$$

Test continuity of g(x) for $x \in [-3, 1]$.

17. If $f(x) = x^2 - 2|x|$ and

$$g(x) = \begin{cases} \min.\{f(t) : -2 \le t \le x\}, & -2 \le x \le 0 \\ \max.\{f(t) : 0 \le t \le x\}, & 0 \le x \le 3 \end{cases}.$$

(a) Draw the graph of f(x) and discuss its continuity and differentiability.

(b) Find and draw the graph of g(x). Also, discuss the continuity.

18. Suppose that f(x) is continuous at a point 'a'. Prove that it is differentiable at 'a' if and only if there exists some linear function L(x) such that $\frac{|f(x) - L(x)|}{|x - a|}$ tends to '0' as x tends to 'a'.

19. Suppose that f(x) is a differentiable function such that $f'(x) = g(x)$, $g''(x)$ exists and $|f(x)| \le 1$ for all real x. If $\{f(0)\}^2 + \{g(0)\}^2 = 9$, then show that for some $c \in (-3, 3)$, $g(c)g''(c) < 0$.

20. If $f(x) = \cos x . \cos\left(\frac{1}{\cos x}\right)$ when $x \in (0, 2\pi) - \{\pi/2, 3\pi/2\}$, and $f(x) = 0$ when $x \in \{\pi/2, 3\pi/2\}$. Show that f is continuous but not differentiable at $x = \pi/2, 3\pi/2$.

21. Prove that the function $f(x) = \begin{cases} e^{-1/x^2} & \text{for } x \ne 0 \\ 0 & \text{for } x = 0 \end{cases}$ is infinitely differentiable at the point $x = 0$.

22. A function $f : R \to [1, \infty)$ satisfies the equation $f(xy) = f(x)\,f(y) - f(x) - f(y) + 2$. If f is differentiable on R and f(2) = 5, then show that $f'(x) = \frac{f(x) - 1}{x} . f'(1)$ Hence, determine f(x).

23. Let $f : R \to R$ satisfy $|f(x)| \le x^2 e^x \;\forall\; x \in R$, then show that f(x) is differentiable at $x = 0$.

24. Let $f(x)=\dfrac{|x-1|}{1+1/x}$ and $g(x)=\dfrac{1}{1+\dfrac{1}{f(x)}}$, then find the points where g(x) is not differentiable.

25. If $f(x+y)=f(x)+f(y)\ \forall\ x,y\in R$, $f(1)=1$ then find $\lim\limits_{x\to0}\dfrac{2^{f(\tan x)}-2^{f(\sin x)}}{x^2 f(\sin x)}$

26. Let $f(x)=\begin{cases} x-1, & -1\le x<0 \\ x^2, & 0\le x\le 1\end{cases}$. Discuss the differentiability of $h(x) = f(|\sin x|) + |f(\sin x)|$ in $[0, 2\pi]$.

27. Let $f: R\to R$ satisfy $f(xy)=f(x).f(y)-f(x+y)+1$ for all x, y ∈ R. If f(x) is differentiable and $f'(0)=1$ then determine all functions 'f'.

28. Suppose that function g(t) and h(t) are defined for all values of t and that g(0) = h(0) = 0. Can $\lim\limits_{t\to0} g(t)/(h(t)$ exist? If it does exist, must it equal zero? Give reason for your answer.

29. Given a function f, we say that f is differentiable at $x_0\in D_f$ if there exists a real number A such that $\lim\limits_{x\to x_0}\dfrac{|f(x)-f(x_0)-A(x-x_0)|}{|x-x_0|}$.
Prove that if f is differentiable at x_0 then the value of A in the above definition is the derivative $f'(x_0)$. (Note that the magnification function $A(x-x_0)$ is called the differential of f at x_0, and A is called the differential coefficient.)

30. A differentiable function f satisfies
$(x-y)\,f(x+y)-(x+y)\,f(x-y)=4xy(x^2-y^2)\ \forall\ x, y\in R$, where $f(1)=1$.
Find f(x).

31. A differentiable function f satisfies the relation $f(x+y)-f(x-y)=f(x).\,f(y)\ \forall\ x, y\in R$.
If $f'(0)=0$, find f(x).

32. Let f be defined on (a, b) and differentiable at c, where a < c < b. Suppose $x_n\to c$ and $z_n\to c$, where
$x_n<c<z_n$ for all n.
Prove that $\dfrac{f(x_n)-f(z_n)}{x_n-z_n}\to f'(c)$.

33. A differentiable function f satisfies the relation
$f\left(\dfrac{x+y}{1+xy}\right)=f(x).\,f(y)\ \forall\ x, y\in R-\{-1\}$, where $f(0)\neq0$ and $f'(0)=1$. Find f(x).

34. A differentiable function f satisfies the relation $[1+f(x).f(y)]\,f(x+y)=f(x)+f(y)\ \forall\ x, y\in R$.
If $f(0)=0$ and $f'(0)=1$, find f(x).

35. Given a differentiable function f(x) defined for all real x, and is such that $f(x+h)-f(x)\le 6h^2$ for all real h and x. Show that f(x) is constant.

Previous Year's Questions (JEE Advanced)

A. Fill in the blanks :

1. Let $f(x)=\begin{cases}(x-1)^2\sin\dfrac{1}{(x-1)}-|x| & \text{if } x\neq1\\ -1 & \text{if } x=1\end{cases}$

be a real-valued functions. Then the set of points where f(x) is not differentiable is........... **[IIT - 1981]**

2. Let $f(x)=x\,|x|$. The set of points where f(x) is twice differentiable is.............. **[IIT - 1992]**

B. Multiple Choice Questions with ONE correct answer :

3. For a real number y, let [y] denotes the greatest integer less than or equal to y : Then the fuction $f(x)=\dfrac{\tan(\pi[x-\pi])}{1+[x]^2}$ is **[IIT - 1981]**

(A) discontinuous at some x
(B) continuous at all x, but the derivative f′(x) does not exist for some x
(C) f′(x) exist for all x, but the derivative f″(x does not exist for some x
(D) f′(x) exits for all x

4. If $f(x)=x(\sqrt{x}-\sqrt{x+1})$, then **[IIT - 1985]**

(A) f(x) is continuous but not differentiable at x = 0
(B) f(x) is differentiable at x = 0
(C) f(x) is not differentiable at x = 0
(D) none of these

5. Let [x] denote the greatest integer less than or equal to x. If $f(x) = [x \sin \pi x]$, then f(x) is **[IIT - 1986]**

(A) continuous at $x = 0$
(B) continuous in $(-1, 0)$
(C) differentiable at $x = 1$
(D) differentiable in $(-1, 1)$
(E) none of these

6. The set of all points where the function $f(x) = \dfrac{x}{(1+|x|)}$ is differentiable, is **[IIT - 1987]**

(A) $(-\infty, \infty)$ (B) $[0, \infty)$
(C) $(-\infty, 0) \cup (0, \infty)$ (D) $(0, \infty)$
(E) none of these

7. Let $f: R \to R$ be a differentiable function and $f(1) = 4$. Then the value of $\lim\limits_{x\to1} \int_4^{f(x)} \dfrac{2t}{x-1} dt$ is **[IIT - 1990]**

(A) $8f'(1)$ (B) $4f'(1)$
(C) $2f'(1)$ (D) $f'(1)$

8. Let [.] denote the greatest integer function and $f(x) = [\tan^2 x]$, then : **[IIT - 1993]**

(A) $\lim\limits_{x\to0} f(x)$ does not exist
(B) f(x) is continuous at $x = 0$
(C) f(x) is not differentiable at $x = 0$
(D) $f'(0) = 1$

9. The function $f(x) = (x^2 - 1)|x^2 - 3x + 2| + \cos(|x|)$ is Not differentiable at - **[IIT - 1999]**

(A) -1 (B) 0
(C) 1 (D) 2

10. Find left hand derivative of $f(x) = [x]\sin(\pi x)$ at k, k an integer if **[IIT - 2001]**

(A) $(-1)^k (k-1)\pi$ (B) $(-1)^{k-1}(k-1)\pi$
(C) $(-1)^k (k-1)k\pi$ (D) $(-1)^{k-1}(k-1)k\pi$

11. Let $f: R \to R$ be a function which is defined by $f(x) = \max\{x, x^3\}$. The set of all points where f(x) is not differentiable is - **[IIT - 2001]**

(A) $\{-1, 1\}$ (B) $\{-1, 0\}$
(C) $\{0, 1\}$ (D) $\{-1, 0, 1\}$

12. Which of the following functions is differentiable at $x = 0$? **[IIT - 2001]**

(A) $\cos(|x|) + |x|$ (B) $\cos(|x|) - |x|$
(C) $\sin(|x|) + |x|$ (D) $\sin(|x|) - |x|$

13. The domain of the derivative of the function

$$f(x) = \begin{cases} \tan^{-1} x, \text{ if } |x| \le 1 \\ \frac{1}{2}(|x|-1), \text{ if } |x| > 1 \end{cases}$$

is **[IIT - 2002]**

(A) $R - \{0\}$ (B) $R - \{1\}$
(C) $R - \{-1\}$ (D) $R - \{-1, 1\}$

14. If f(x) is differentiable and strictly increasing function, then the value of $\lim\limits_{x\to0} \dfrac{f(x^2) - f(x)}{f(x) - f(0)}$ is **[IIT - 2004]**

(A) 1 (B) 0
(C) -1 (D) 2

15. The function given by $y = ||x| - 1|$ is differentiable for all real number except the points **[IIT - 2004]**

(A) $\{0, 1, -1\}$ (B) ± 1
(C) 1 (D) -1

16. If f is a differentiable function and $f(1/n) = 0 \ \forall \ n \ge 1$ and $n \in I$, then **[IIT - 2005]**

(A) $f(x) = 0, x \in (0, 1]$
(B) $f(0) = 0, f'(0) = 0$
(C) $f(0) = 0 = f'(x), x \in (0, 1]$
(D) $f(0) = 0$ and $f'(0)$ need not to be zero

17. Let f(x) be differentiable on the interval $(0, \infty)$ such $f(1) = 1$, and $\lim\limits_{t\to x} \dfrac{t^2 f(x) - x^2 f(t)}{t - x} = 1$ for each $x > 0$. Then f(x) is **[IIT - 2007]**

(A) $\dfrac{1}{3x} + \dfrac{2x^2}{3}$ (B) $\dfrac{-1}{3x} + \dfrac{4x^2}{3}$
(C) $\dfrac{-1}{x} + \dfrac{2}{x^2}$ (D) $\dfrac{1}{x}$

18. Let $g(x) = \dfrac{(x-1)^n}{\log \cos^m (x-1)}$; $0 < 2$, m and n are integers, $m \ne 0$, $n > 0$ and let p be the left hand derivative of $|x - 1|$ at $x = 1$. If $\lim\limits_{x\to1} g(x) = p$, then : **[IIT - 2008]**

(A) $n = 1\ n = 1$ (B) $n = 1, m = -1$
(C) $n = 2, m = 2$ (D) $n > 2, m = n$

19. Let $f(x) = \begin{cases} x^2 \left|\cos\dfrac{\pi}{x}\right|, x \ne 0 \\ 0, x = 0 \end{cases}, x \in,$ then f is

(A) differentiable both at $x = 0$ and at $x = 2$
(B) differentiable at $x = 0$ but not differentiable at $x = 2$
(C) not differentiable at $x = 0$ but differentiable at $x = 2$
(D) differentiable neither at $x = 0$ nor at $x = 2$

[IIT - 2012]

C. Multiple Choice Questions with ONE or MORE THAN ONE correct answer :

20. If $x + |y| = 2y$, then y as a function of x is **[IIT - 1984]**

(A) defined for all real x
(B) continuous at $x - 0$
(C) differentiable for all x
(D) such that $\frac{dy}{dx} = \frac{1}{3}$ for $x < 0$

21. The function $f(x) = 1 + |\sin x|$ is **[IIT - 1986]**

(A) continuous now here
(B) continuous everywhere
(C) differentiable nowhere
(D) not differentiable at $x = 0$
(E) not differentiable at infinite number of points.

22. The function $f(x) = \begin{cases} |x-3|, & x \geq 1 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}, & x < \end{cases}$ is **[IIT - 1988]**

(A) continuous at $x = 1$
(B) differentiable at $x = 1$
(C) continuous at $x = 3$
(D) differentiable at $x = 3$

23. Let $h(x) = \min\{x, x^2\}$, for every real number of x. Then **[IIT - 1998]**

(A) h is continuous for all x
(B) h is differentiable for all x
(C) $h'(x) = 1$, for all $x > 1$
(D) h is not differentiable at two values of x'

24. Let $f(x) = \begin{cases} 0, & x < 0 \\ x^2, & x \geq 0 \end{cases}$ then for all x **[IIT - 1994]**

(A) f' is differentiable (B) f is differentiable
(C) f' is continuous (D) f is continuous

25. Let $g(x) = x\,f(x)$, where $f(x) = \begin{cases} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$
At $x = 0$ **[IIT - 1994]**

(A) g is differentiable but g' is not continuous
(B) g is differentiable while f is not
(C) both f and g are differentiable
(D) g is differentiable and g' is continuous

26. The function $f(x) = \max\{(1 - x), (1 + x), 2\}$, $x \in (-\infty, \infty)$ is **[IIT - 1995]**

(A) continuous at all points
(B) differentiable at all points
(C) differentiable at all points except at $x = 1$ and $x = -1$
(D) continuous at all points except at $x = 1$ and $x = -1$, where it is discontinuous

27. Let $f : R \to R$ be any function. Define $g : R \to R$ by $g(x) = |f(x)|$ for all x. Then g is - **[IIT - 2000]**

(A) onto if f is onto
(B) one-one if f is one-one
(C) continuous if f is continuous
(D) differentiable if f is differentiable

28. If $f(x) = \min.\{1, x^2, x^3\}$, then **[IIT - 2006]**

(A) f(x) is continuous $\forall\ x \in R$
(B) f(x) is continuous & differentiable everywhere
(C) f(x) is continuous but not differentiable $\forall\ x \in R$
(D) f(x) is not differentiable at two points

29. If $f(x) = \begin{cases} -x - \frac{\pi}{2}, & x \leq -\frac{\pi}{2} \\ -\cos x, & -\frac{\pi}{2} < x \leq 0 \\ x - 1 & 0 < x \leq 1 \\ \ell n\, x, & x > 1 \end{cases}$, then **[IIT - 2011]**

(A) f(x) is continuous at $x = -\frac{\pi}{2}$
(B) f(x) is not differentiable at $x = 0$
(C) f(x) is differentiable at $x = 1$
(D) f(x) is differentiable at $x = -\frac{3}{2}$

30. Let $f : R \to R$ be a function such that $f(x + y) = f(x) + f(y)$, $\forall\ x, y \in R$. If f(x) is differentiable at $x = 0$, then **[IIT - 2011]**

(A) f(x) is differentiable only in a finite interval containing zero
(B) f(x) is continuous $\forall\ x \in R$
(C) $f'(x)$ is constant $\forall\ x \in R$
(D) f(x) is differentiable except at finitely many points

31. Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that
$$f(x+y)=f(x)+f(y),\ \forall\ x,y\in\mathbb{R}$$
If f(x) is differentialbe at x = 0, then **[IIT - 2011]**
(A) f(x) is differentiable only in a finite interval containing zero
(B) f(x) is continuous $\forall\ x\in\mathbb{R}$
(C) f '(x) is constant $\forall\ x\in\mathbb{R}$
(D) f(X) is differentiable except at finitely many points.

32. If $f(x)=\begin{cases}-x-\frac{\pi}{2}, x\le-\frac{\pi}{2}\\ -\cos x, -\frac{\pi}{2}<x\le 0\\ x-1, 0<x\le 1\\ \ln x, x>1\end{cases}$, then

[IIT - 2011]
(A) f(x) is continuous at $x=-\frac{\pi}{2}$
(B) f(x) is not differentiable at x = 0
(C) f(x) is differentiable at x = 1
(D) f(x) is differentiable at $x=-\frac{3}{2}$

D. Assertion & Reasoning :

33. Let f and g be real valued functions defined on interval (–1, 1) such that g″(x) is continuous, $g(0)\neq 0$, $f''(0)=0$ and $f(x)=g(x)\sin x$.
Statement–1 : $\lim_{x\to 0}[g(x)\cot x-g(0)\operatorname{cosec} x]=f''(0)$
Statement–2 : $f'(0)=g(0)$ **[IIT - 2008]**

E. Integer Answer Type :

34. If the function $f(x)=x^3+e^{x/2}$ and $g(x)=f^{-1}(x)$, then the value of $g'(1)$ is : **[IIT - 2009]**

35. Let $y'(x)+y(x)\,g'(x)=g(x)$, $y(0)=0$, $x\in\mathbb{R}$, where f'(x) denotes $\frac{df(x)}{dx}$ and g(x) is a given non-constant differentiable function on $\mathbb{R}$ with g(0) = g(2) = 0. Then the value of y(2) is **[IIT - 2011]**

F. Subjective Problems :

36. Let $f(x)=\frac{x^2}{2}, 0\le x<1$ **[IIT - 1983]**
$$=2x^2-3x+\frac{3}{2}, 1\le x\le 2$$
Discuss the continuity of f, f′ and f″ on [0, 2].

37. Let $f(x)=x^3-x^2+x+1$ and **[IIT - 1985]**
$g(x)=\max\{f(t); 0\le t\le x\}, 0\le x\le 1$
$=3-x \qquad 1<x\le 2$
Discuss the continuity and differentiability of the function g(x) in the interval (0, 2).

38. Let f(x) be defined in the interval [–2, 2] such that
$f(x)=1, -2\le x\le 0$
$=x-1, 0<x\le 2$
and $g(x)=f(|x|)+|f(x)|$
Test the differentiability of g(x) on $x\in[-2,2]$
[IIT - 1986]

39. Let f(x) be a function satisfying the condition f(–x) = f(x) for all real x. If f'(0) exists, find its value.
[IIT - 1987]

40. Draw a graph of the function y = [x] + |1 – x|, $-1\le x\le 3$. Determine the points, if any, where this function is not differentiable. **[IIT - 1989]**

41. A function R → R satisfies the equation f(x + y) = f(x) f(y) for all x, y in R and $f(x)\neq 0$ for any x in R. Let the function be differentiable at x = 0 and f'(0) = 2. Show that f'(x) = 2 f(x) for all x in R. Hence, determine f(x). **[IIT - 1990]**

42. Let $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$ for all real x and y. If f′(0) exists and equals – 1 and f(0) = 1. Find f(2)
[IIT - 1995]

43. Let $f(x)=\begin{cases}xe^{\left(-\frac{1}{|x|}+\frac{1}{x}\right)}; & x\neq 0\\ 0 \quad ; & x\neq 0\end{cases}$ test whether
[IIT - 1997]
(a) f(x) is continuous at x = 0
(b) f(x) is differentiable at x = 0

44. Determine the values of x for which the following function fails to be continuous or differentiable
$$f(x)=\begin{cases}1-x & ;\ x<1\\ (1-x)(2-x) & ;\ 1\le x\le 2\\ 3-x & ;\ x>2\end{cases}$$
Justify your answer. **[IIT - 1997]**

45. Discuss the continuity and differentiability of the function **[IIT - 1998]**

$$f(x)=\begin{cases}2+\sqrt{1-x^2} & ; |x|\le 1\\ 2e^{(1-x)^2} & ; |x|>1\end{cases}$$

46. Let $\alpha \in R$. Prove that a function $f: R \to R$ is differentiable at α if and only if there is a function $g: R \to R$ which is continuous at α and satisfies $f(x)-f(\alpha)=g(x)(x-\alpha)$ for all $x \in R$. **[IIT - 2001]**

47. Let $f(x)=\begin{cases}x+a & \text{if } x<0\\ |x-1| & \text{if } x\ge 0\end{cases}$, and

$g(x)=\begin{cases}x+1 & \text{if } x<0\\ (x-1)^2+b & \text{if } x\ge 0\end{cases}$, where a and b are non-negative real numbers Determine the composite function g of. If (g o f) (x) is continuous for all real x, determine the values of a and b. Further, for these value of a and b, is gof differentiable at x = 0 ? Justify your answer. **[IIT - 2002]**

48. If a function $f:[-2a, 2a] \to R$, is an odd function such that $f(x) = f(2a - x)$ for $x \in [a, 2a]$ and left hand derivative at x = a is 0, then find the left hand derivative at x = – a. **[IIT - 2003]**

49. Let $f(x)=\begin{cases}b\sin^{-1}\dfrac{x+c}{2} & ; -1/2<x<0\\ 1/2 & ; x=0\\ \dfrac{e^{ax/2}-1}{x} & ; 0<x<\dfrac{1}{2}\end{cases}$

If f(x) be a differentiable function at x = 0, & $|c| < 1/2$, then find the value of 'a' and prove that $64b^2=(4-c^2)$. **[IIT - 2004]**

50. If $f(x-y)=f(x).g(y)-f(y).g(x)$ and $g(x-y)=g(x).g(y)+f(x).f(y)$ for all $x, y \in R$. If right hand derivative at x = 0 exists for f(x). Find derivative of g(x) at x = 0 **[IIT - 2005]**

G. Match the Columns

51.

Column I	Column II
(A) $x\|x\|$	**(P)** continuous in (–1, 1)
(B) $\sqrt{\|x\|}$	**(Q)** differentiable in (–1, 1)
(C) $x+[x]$	**(R)** strictly increasing in (–1, 1)
(D) $\|x-1\|+\|x+1\|$	**(S)** not differentiable at least at one point in (–1, 1)

ANSWERS

CONCEPT PROBLEMS—A

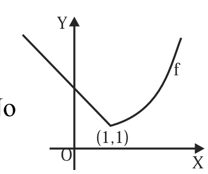
1. $f'(0^+)=-1$; $f'(0^-)=1$; non-differentiable
2. $m=-1$, $n=\pi$.
3. $f'(1+) = -1$; $f'(1-) = 3\ln 3$ hence f is not differentiable at x = 1
4. (b) $(2a^{-1/3})/3$
5. yes; y = 1.
6. non-differentiable at x = 0

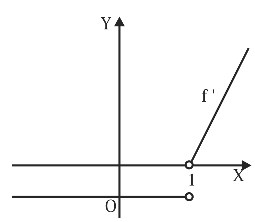
PRACTICE PROBLEMS—A

7. – h; 0.
8. (a) –1,1 (b) $\dfrac{2}{a}, \dfrac{-2}{a}$ (c) 1, 0 (d) 0, 0.
9. f(x) is not differentiable at x = 0
14. 0
15. 0

CONCEPT PROBLEMS—B

1. (a) –1, 2 (b) 1, 3
2. (a) 0 (b) 1, 3
3. (a) 5 (b) 2, 3
4. – 4 (discontinuity), – 1 (corner), 2 (discontinuity), 5 (vertical tangent).
5. No





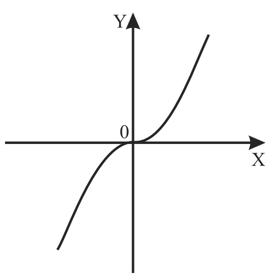
7. $a = 2, b = 0$
8. continuous and differentiable.
9. cont. but not diff.at $x = 0$; diff. & cont. at $x = \pi/2$
10. $a = 1, b = 1$
11. discontinuous and non-differentiable at $x = 1$.

PRACTICE PROBLEMS—B

12. continuous; diff at $x = 1$, non diff. at $x = 0$
13. continuous but non diff at $x = 0$
14. $a = 2$
15. diff.
16. discontinuous & not derivable at $x = 1$, continuous but not derivable at $x = 2$
18. discontinuous at $x = 1$ and hence non-differentiable.

CONCEPT PROBLEMS—C

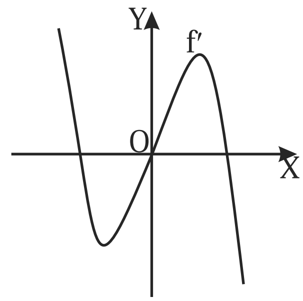
1. (a) $x = -3, 1$ (b) $x = \pm 1$.
2. yes
3. (a) $-1, 1$
   (b) $x = 10/3$, L.H.D. $= -3$, R.H.D. $= 3$
5. $f'(o)$ does not $f'(o^-) = 0$ $f'(o^+)$
6. $f(x)$ is cont. but not diff. at $x = 0$
7. (a)



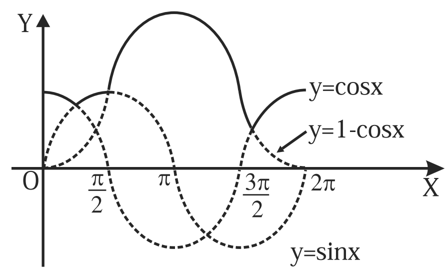
(b) All x (c) $f'(x) = 2|x|$
8. discontinuous at $x = 0, 1, 2, 3$ hence non-differentiable
9. The Dirichlet function is nowhere continuous, and hence it is nowhere derivable.
10. diff every where.

PRACTICE PROBLEMS—C

11. continuous but non-differentiable.
12. differentiable.
13. $x = 0, \pm 1$
14. $a = 35/9, b = 10/3$
15. Non differentiable.
16.



17. f is cont. but not diff. at $x = 1$, discont. at $x = 2$ & $x = 3$. cont.& diff. at all other points
18. 3 ; The bold line represents the graph of $y = f(x)$.



It is non-differentiable at three points namely

$$x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{3}.$$

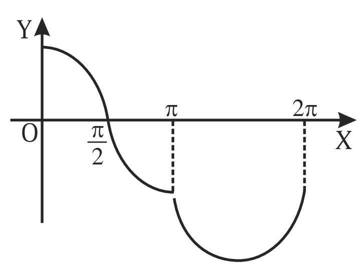
19. $g \circ f(x) = 0$ for all $x \in R$ and

$$f \circ g = \begin{cases} 0 \text{ if } x \text{ is an integer} \\ n \text{ if } x \in R^+ \text{and } \sqrt{n} < x < \sqrt{n+1} \text{ or} \\ x \in R^- \text{and } -\sqrt{n+1} < x < -\sqrt{n} \end{cases}$$

g o f is differentiable everywhere and f o g is differentiable on $R - I$.

20. $$g(x) = \begin{cases} \cos x, & 0 \le x \le \pi \\ \sin x - 1, & x > \pi \end{cases}$$

Adjacent figure represents the graph of $g(x)$. Clearly, $g(x)$ is continuous but non-differentiable at $x = \pi$.



CONCEPT PROBLEMS—D

1. 0

2. $f(x) = x^6$, $a = 2$

3. (i) $f(x) = 2^x$, $a = 5$
(ii) $f(x) = \cos x$, $a = \pi$ or $f(x) = \cos(\pi + x)$, $a = 0$

4. The only doubtful point is 0. But $|f(x)/x| = |x| \to 0$ as $x \to 0$, so $f'(0) = 0$. For $x > 0$, $f'(x) = 2x$ and for $x < 0$, $f'(x) = -2x$. therefore $f'(x) = 2|x|$ is continuous.

5. 1

6. 2/3

7. non diff. at $x = 0$

8. $a = 6$, $b = -3$.

CONCEPT PROBLEMS—E

2. (a) x for $x \ge 0$, $-x$ for $x < 0$
(b) Yes
(c) 1 for $x > 0$, -1 for $x < 0$

3. (a) $\frac{4}{3}x^{1/3}$ (b) Yes
(c) $\frac{4}{9x^{2/3}}$

4. (a) $g'(x) = 2x$ for $x \le 1$, $g'(x) = 2$ for $x > 1$
$g''(x) = 2$ for $x \le 1$, $g''(x) = 0$ for $x > 1$
(b) g is differentiable ; g' is not

5. $-\sin a$

PRACTICE PROBLEMS—D

6. (a) $y'' = 6|x|$; $y''(0) = 0$.
(b) $y'' = 2\sin\frac{1}{x} - \frac{2}{x}\cos\frac{1}{x} - \frac{1}{x^2}\sin\frac{1}{x}$ at $x \ne 0$, $y''(0)$ does not exist, since $y'(x)$ is discontinuous at $x = 0$.

8. $2\sec^x x \tan x$

9. $n - 1 < k < n$

CONCEPT PROBLEMS—F

1. (a) Yes (b) No

2. No

3. (a) No (b) No

4. If $g(x_0) \ne 0$ then f is non-diff. as $x = x_0$. If $g(x_0) = 0$ then f is differentiable is $x = x_0$.

PRACTICE PROBLEMS—E

6. Yes

7. No

8. (b) f is twice but not thrice differentiable at $x = x_0$ and the derivatives are 0.

9. They are of opposite signs.

10. non-differentiable.

11. Cont. but non diff. at $x = 1$, discont. and non diff. at $x = 2, 3$

12. non diff. at $x = \pi, 2\pi$

13. $p > 1$

14. non-diff at $x = 2, 3$ if $a \ne 4, 9$; non diff at $x = 3$ if $a = 4$; non-diff at $x = 2$ if $a = 9$.

PRACTICE PROBLEMS—F

2. $x, |x|$

3. -1

4. 0

5. $2x$

7. $\frac{1}{2}, \ell n 2$

8. $x^2 + x e^x$

10. $f(x) = 2x$

OBJECTIVE EXERCISES

1. C	**2.** B	**3.** A
4. A	**5.** B	**6.** B
7. D	**8.** D	**9.** B
10. B	**11.** A	**12.** D
13. A	**14.** A	**15.** A
16. B	**17.** A	**18.** D
19. A	**20.** B	**21.** C
22. C	**23.** B	**24.** D
25. C	**26.** B	**27.** A
28. C	**29.** C	**30.** C
31. D	**32.** A	**33.** C
34. C	**35.** C	**36.** A
37. D	**38.** D	**39.** A
40. A	**41.** C	**42.** D
43. A	**44.** C	**45.** A
46. B	**47.** D	**48.** D
49. A	**50.** C	**51.** A,C

52. A,D **53.** A,B,C **54.** B,D
55. A,C **56.** A,B,C **57.** A,B,D
58. B,D **59.** A,B,C **60.** B,C,D
61. B,D **62.** A,B,D **63.** A,B,C,D
64. A,C,D **65.** A,B,D **66.** A,B
67. A,B **68.** A,D **69.** C,D
70. B,C **71.** A **72.** A
73. D **74.** A **75.** D
76. A **77.** D **78.** B
79. C **80.** B **81.** A
82. C **83.** B **84.** D
85. A **86.** A **87.** B
88. D **89.** B **90.** D
91. C **92.** D **93.** C
94. C **95.** B
96. (A)–(P,Q), (B)–(S), (C)–(Q), (D)–(R)
97. (A)–R,(B)–S, (C)–P, (D)–Q
98. (A)–(P), (B)–(R), (C)–(Q), (D)–(T)
99. (A)–P; (B)–(P, Q), (C)–(P, R), (D)–(R)
100. (A)–(P), B–(R), (C)–(S), (D)–(P)

REVIEW EXERCISES for JEE ADVANCED

1. continuous but non-differentiable.

2. $g(x)=x$

3. discont and non-diff at $x = 0, 2$; cont but non-diff at $x = \pm 1$.

4. discont and non-diff at $x = 0$; cont but non-diff at $x = \pm\sqrt{2}$

5. $\{1, 3\}$

6. non-diff at $x = 2, 4$

7. discontinuous and non-differentiable at $x = 2$.

8. $a = -1, b = -2$, value $= 2/3$.

9. If $f(x_0) = g(x_0)$, then the two functions may or may not be differentiable at $x = 0$.

12. f is cont. but not derivable at $x = x_0$.

14. continuous but non-differentiable at $x = 0$.

15. 180

16. $a \in (0, 1]$

17. If $a \in (0, 1), f'(0^+) = -1, f'(0^-) = 1$

$\Rightarrow$ continuous but not derivable ; If $a = 1$, $f(x) = 0$ which is constant $\Rightarrow$ continuous and derivable;

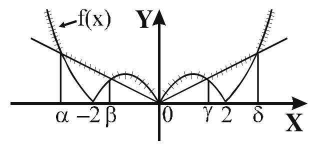
If $a > 1, f'(0^-) = -1, f'(0^+) = 1$

$\Rightarrow$ continuous but not derivable.

18. non diff. at $x = 1$

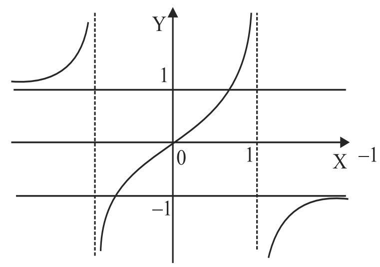
19. $g'(0), 0, g'(4), g'(2), g'(-2)$

20. f(x) is non differentiable at $x = \alpha, \beta, 0, \gamma, \delta$ and g(x) is non differentiable at $x = \alpha, \beta, 0, -2, 2$



21. $f(x) = \begin{cases} \dfrac{x}{1-x} & \text{if } x \geq 0,\ x \neq 1 \\ \dfrac{x}{1+x} & \text{if } x < 0,\ x \neq -1 \end{cases}$

and $f'(x) = \begin{cases} \dfrac{1}{(1-x)^2} & \text{if } x > 0,\ x \neq 1 \\ \dfrac{1}{(1+x)^2} & \text{if } x < 0,\ x \neq -1 \end{cases}$



23. $f'(1^+) = 3,\ f'(1^-) = -1$

25. (a) 2 (b) 1

26. f(x) is not cont. not diff, xf(x) is cont. diff. $x^2f(x)$ is cont and diff.

27. cont but not diff at $x = 1$, discont. at $x = 2, 3$

28. $a \neq 1, b = 0,\ p = 1/3, q = -1$

29. cont in $0 \leq x \leq 1$, but not diff. at $x = 0$

30. $D^*[f(x). g(x)] = f^2(x)\, D^*g(x) + g^2(x).\, D^*f(x)$;

$$D^*\frac{f(x)}{g(x)} = \frac{g^2(x).D^*f(x) - f^2(x).D^*g(x)}{[g(x)]^4}$$

31. $x+1$

32. $f(x)=0$

33. $f(x)=\ln x$

34. $f(x)=1, f(x)=0.$

35. $f(x)=e^{-x}-1$

TARGET EXERCISES for JEE ADVANCED

1. Continuous but non-differentiable at $x=-2, -1, 0, 1, 2$

2. not differentiable at $x=e$; $f'(e^+)=\infty$, $f'(e^-)=0$.

4. $f(x)=0, f(x)=1/2, f(x)=x.$ **5.** $f(x)=\sin^{-1} x)$

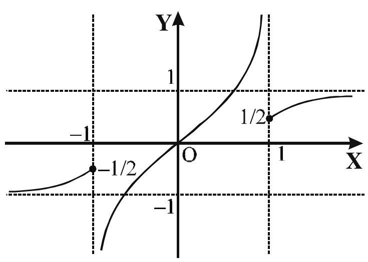
6. continuous but non-differentiable at $x=1$.

7. f is continuous and differentiable for all $x \in$ domain, except non-differentiable at $x=1$

10. 256

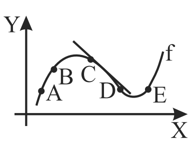
11. 54

13. $R-\{-1, 1\}$



14. (i) A and B (ii) Greater

(iii)



(iv) B and C, D and E

16. g(x) is continuous in $[-3, 1]$

17. (a) continuous for all $x \in R$ not differentiable for all $x \in R-\{0\}$

(b) g(x) is not continuous at $x=0$

22. $1+x^2$ **24.** $\{\pm 1, 0, 1-\sqrt{2}\})$

25. $\frac{\ell n 2}{2}$ **26.** Non - diff. at $x=\pi, 2\pi$

27. $x+1$

28. For $g(t)=mt$ and $h(t)=t$, $\text{limit}_{t\to 0}\, g(t)/(h(t)=m$, which need not be zero.

30. $f(x)=x^3$ **31.** $f(x)=0$

33. $f(x)=\frac{1-x}{1+x}$. **34.** $f(x)=\frac{e^{2x}-1}{e^{2x}+1}$.

PREVIOUS YEAR'S QUESTIONS (JEE ADVANCED)

1. $\{0\}$ **2.** $R-\{0\}$

3. D **4.** B

5. ABD **6.** A

7. A **8.** B

9. D **10.** B

11. D **12.** D

13. D **14.** C

15. A **16.** B

17. A **18.** C

19. B **20.** ABD

21. BDE **22.** ABC

23. ACD **24.** BCD

25. AB **26.** AC

27. C **28.** AC

29. ABCD **30.** BC

31. BC **32.** ABCD

33. B **34.** 2

35. 0

36. f and f' are continuous and f'' is discontinuous on $[0,2]$

37. cont. on $(0, 2)$ and differentiable on $(0, 2)=\{1\}$

38. not differentiable at $x=0, 1$

39. $f'(0)=0$ **40.** $x=0, 1, 2, 3$

41. $f(x)=e^{2x}$ **42.** -1

43. cont. but not diff.

44. f(x) is not continuous and thus not differentiable at $x=2$

45. continuous but not differentiable at discontinuous and not differentiable at $x=-1$.

47. $g(f(x)) = \begin{cases} x+a+1 & \text{if } x < -a \\ (x+a+1)^2 + b & \text{if } a \le x < 0 \\ x^2 + b & \text{if } 0 \le x \le 1 \\ (x-2)^2 + b & \text{if } x > 1 \end{cases}$ $a = 1, b = 0,$

gof differentiable at $x = 0$

48. 0

49. $a = 1$

50. 0

51. (A)$\rightarrow$(P, Q, R), (B)$\rightarrow$(P, S), (C)$\rightarrow$(R, S), (D)$\rightarrow$(P, Q)